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ANALYTICAL METHODS IN SEARCH THEORY

Marc S. Mangel
James A. Thomas, Jr.

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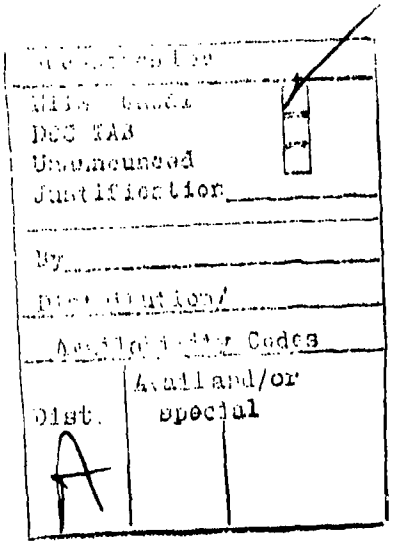

Operations Evaluation Group

CENTER FOR NAVAL ANALYSES

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PREFACE

This work is an expanded version of a set of lecture notes corresponding to a course given by the author during the period November 1978 to January 1979. Analytical methods for the solution of moving object search problems are developed from "first principles." This work complements the first author's CNA memorandum (reference 1) but otherwise there is little overlap with the existing literature on search theory (e.g., references 2, 3, and 4). The approach taken is tutorial, in that the solutions of harder problems are motivated by the solutions of simpler problems. There are exercises interspersed throughout the work; the solutions of some of these are given in the appendix.

This work is concerned with mathematical analysis and not with modeling target motion or detection functions and not with the development of computer codes.

Some general references are references 2 through 7.

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FORMULATION OF THE SEARCH PROBLEM

The position of the target at time t will be denoted by $X(t)$. It is a vector, so that:

$$X(t) = (X_1(t), X_2(t), X_3(t)) \quad . \tag{1.1}$$

The target moves in some region, denoted by D_T.

The position of the searcher at time t will be denoted by:

$$Z(t) = (Z_1(t), Z_2(t), Z_3(t)) \quad . \tag{1.2}$$

The following inputs are needed to formulate the search problem.

INITIAL DENSITY

Let $\rho_0(x)$ be defined by:

$$\rho_0(x)dx = \Pr\{x_1 \le X_1(0) \le x_1 + dx_1,\ x_2 \le X_2(0) \le x_2 + dx_2,\ x_3 \le X_3(0) \le x_3 + dx_3\} \quad . \tag{1.3}$$

In this equation, $x = (x_1, x_2, x_3)$ is a vector and $dx = (dx_1, dx_2, dx_3)$. Three types of initial densities are considered.

Type 1

Initial density concentrated at a point. Namely:

$$\text{Prob}(X(0) = x_0) = 1 . \tag{1-4}$$

Symbolically, one writes:

$$\rho_0(x) = \delta(x-x_0) = \delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30}) . \tag{1-5}$$

In this equation, $\delta(S)$ is the "Dirac Delta Function" (reference 8).

Exercise (See Reference 8)

One definition of $\delta(S)$ is:

$$\delta(S) = \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}}\, e^{-S^2 n/2} \tag{1-6}$$

$$= \lim_{n\to\infty} \delta_n(S) . \tag{1-7}$$

Sketch a few of the $\delta_n(S)$. Also draw $\delta'(S)$, the derivative of $\delta(S)$. Note that $\delta(S) = 0$ if $S \neq 0$ and:

$$\int \delta(S)\,dS = 1. \tag{1-8}$$

$$\int h(S)\delta(S)\,dS = h(0) . \tag{1-9}$$

Type 2: Nowhere Vanishing Initial Densities

In this case, $\rho_0(x)$ does not vanish in D_T. An example of such an initial density is the circular Gaussian:

$$\rho_0(x) = \left[\frac{1}{2\pi\sigma^2}\right]^{3/2} e^{-r^2/2\sigma^2} , \tag{1-10}$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Type 3: Compact Initial Data

In this case, $\rho_0(x)$ is non-zero in some subregion $D_T' \subseteq D_T$ and zero elsewhere (see figure 1).

TARGET MOTION MODEL

The velocity of the target is $V(X,t)$ and is given by:

$$\frac{dX}{dt} = V(X,t) . \tag{1-11}$$

The increment in X is then given by $\Delta X = V(X,t)\Delta t$. Introduce the transition function (e.g., references 5 and 6), $q(\xi,t,\Delta t,x)$ defined as follows.

Set:

$$\Delta X = X(t+\Delta t) - X(t) . \tag{1-12}$$

Then, the transition function $q(\xi,t,\Delta t,x)$ is:

$$q(\xi,t,\Delta t,x)d\xi = \text{Prob}\{\xi \le \Delta X \le \xi + d\xi \,|\, X(t)=x\} . \tag{1-13}$$

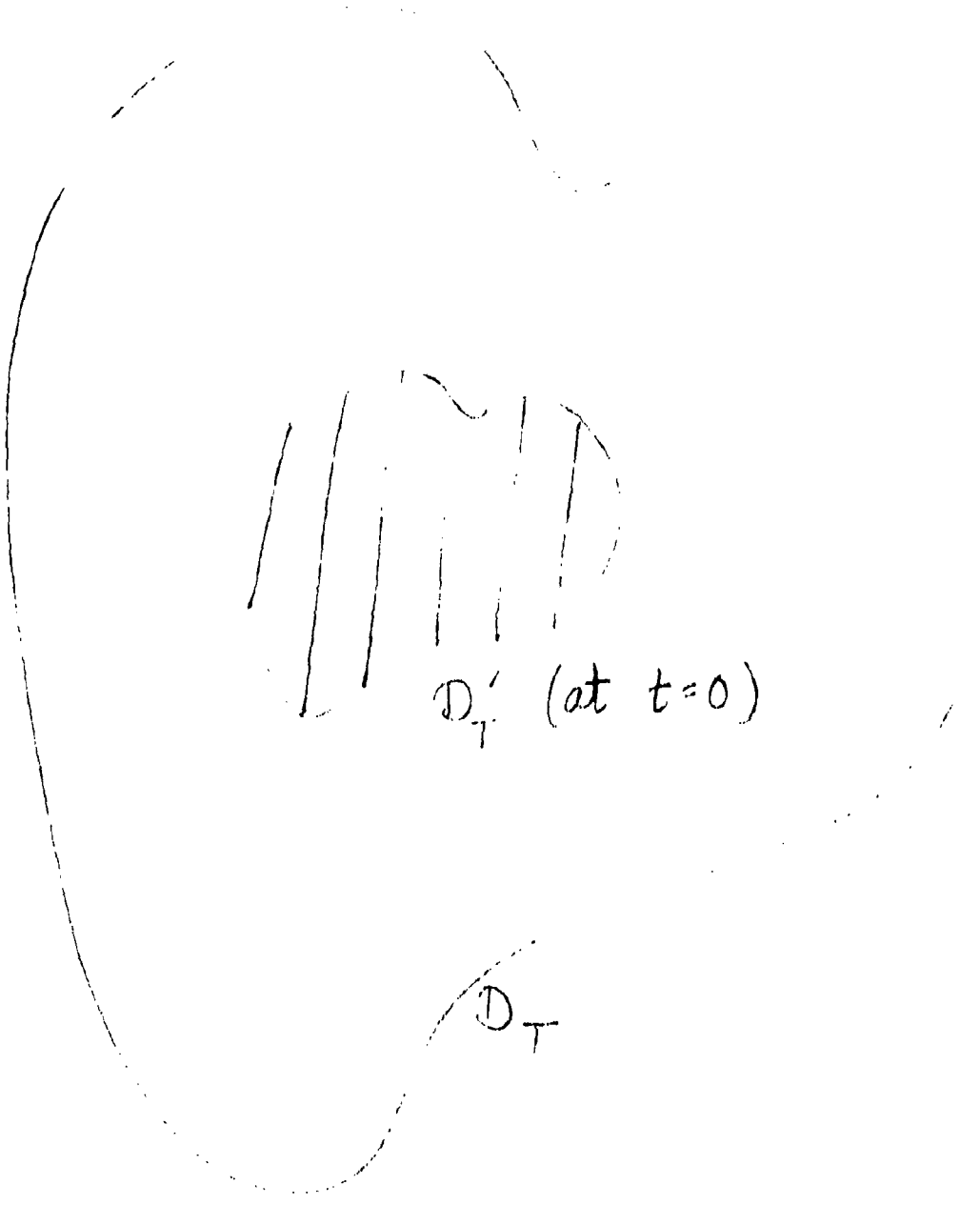

FIG. 1: ILLUSTRATION OF D_T AND TYPE 3 INITIAL DATA

In this equation, $\xi = (\xi_1, \xi_2, \xi_3)$ and $d\xi = (d\xi_1, d\xi_2, d\xi_3)$. The right-hand side of equation 1-13 is satisfied by each component. Equation 1-13 can also be rewritten as:

$$q(\xi,t,\Delta t,x)d\xi = \text{Prob}\{\xi \le X(t+\Delta t) - X(t) \le \xi+d\xi \,|\, X(t) = x\}. \quad (1\text{-}14)$$

Deterministic Target Motion

In the case of deterministic target motion,

$$q(\xi,t,\Delta t,x) = \delta(\xi - b(x,t)\Delta t) \quad . \quad (1\text{-}15)$$

Namely, with probability 1, $\Delta X = b(x,t)\Delta t$. Let $E\{\cdot\}$ denote mathematical expectation. Then:

$$\begin{aligned} E\{\Delta X \,|\, X(t) = x\} &= \int \xi q(\xi,t,\Delta t,x)d\xi \\ &= \int \xi \delta(\xi - b(x,t)\Delta t)d\xi \quad (1\text{-}16) \\ &= b(x,t)\Delta t. \end{aligned}$$

Hence, one writes that:

$$\frac{dX}{dt} = b(X,t). \quad (1\text{-}17)$$

Exercise

Show that if $q(\xi,t,\Delta t,x)$ is given by equation 1-16, then:

$$\mathrm{Var}[\Delta X | X(t) = x] = \quad .$$

Conditionally Deterministic Target Motion

In this case, equation 1-16 is replaced by:

$$q(\xi,t,\Delta t,x) = \sum_{\alpha} p_{\alpha}\delta(\xi - b^{\alpha}(x,t)\Delta t) \ , \qquad (1\text{-}18)$$

with $\sum_{\alpha} p_{\alpha} = 1$. Namely, there are a number of possible target velocities, with a probability p_{α} associated with velocity $b^{\alpha}(x,t)$. If the velocities are continuously distributed, then equation 1-18 must be modified, of course.

The classical example of conditionally deterministic target motion is the "fleeing datum" (see reference 2). In this case, the target flees with a known velocity but with unknown bearing.

Stochastic Target Motion

In this case, one assumes that given $X(t) = x$, then $\Delta X = X(t+\Delta t) - X(t)$ is normally distributed with mean $\hat{b}(x,t)\Delta t + o(\Delta t)$*

*Without loss of generality, one assumes $\hat{b}(x,t)$ is known with complete certainty. Only a small modification is needed to use a conditionally deterministic mean $\hat{b}^{\alpha}(x,t)$.

and covariance matrix $(\hat{a}_{ij}(x,t)\Delta t + o(\Delta t))$. Here $o(\Delta t)$ means a quantity such that:

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0 \quad . \tag{1-19}$$

Alternatively, one writes that:

$$E\{\Delta X_i | X(t) = x\} = \hat{b}_i(x,t)\Delta t + o(\Delta t) \qquad i = 1,2,3 \tag{1-20}$$

$$Cov\{\Delta X_i \Delta X_j | X(t) = x\} = \hat{a}_{ij}(x,t)\Delta t + o(\Delta t) \quad . \tag{1-21}$$

Exercise

Show that:

$$E\{\Delta X_i \Delta X_j | X(t) = x\} = \hat{a}_{ij}(x,t)\Delta t + o(\Delta t) \tag{1-22}$$

Exercise

Let $\Delta t \to 0$ and then $(\hat{a}_{ij}) \to 0$. What happens to the density $q(\xi,t,\Delta t,x)$ for ΔX?

Exercise

Restate equations 1-20 and 1-22 in terms of $q(\xi,t,\Delta t,x)$.

Remark:

1. Since for stochastic target motion

$$dX \sim N(\hat{b}(x,t)\Delta t + o(\Delta t),\ \hat{a}(x,t)\Delta t + o(\Delta t)) \ , \quad X(t)$$

satisfies the "Ito Equation" $dX = b(X,t)dt + \sqrt{a(X,t)}\ dW$, where $W(t)$ is Brownian motion.

2. For the case of stochastic target motion

$$\int \xi^n q(\xi,t,\Delta t,x)d\xi \ = \ o(\Delta t) \quad \text{for} \quad n \geq 3 \ .$$

Exercise

Demonstrate Remark 2.

SEARCH FUNCTION

The final input is the search function (or "conditional detection function", reference 9). Let:

$$\hat{\psi}(x,t,z)\Delta t \ = \ \text{Prob}\{\text{detection in } (t,\ t+\Delta t) \mid X(t)=x,\ Z(t)=z\}. \qquad (1\text{-}23)$$

This function must be modeled.

SEARCH PROBLEMS

The following quantities are of interest in search theory.

Let:

$$\rho(x,t|Z)dx = \Pr\{x \le X(t) < x + dx | \text{search along } Z(\tau), \; 0 \le \tau \le t \text{ was not successful}\} \quad . \tag{1-24}$$

To obtain $\rho(x,t|Z)$, consider the joint density

$$f(x,t;Z)dx = \Pr\{x \le X(t) \le x + dx \text{ and search along } Z(\tau), \; 0 \le \tau \le t \text{ was not successful}\} \quad . \tag{1-25}$$

Since $\Pr(A \text{ and } B) = \Pr(A|B)\Pr(B)$, one has:

$$\rho(x,t|Z) = \frac{f(x,t;Z)}{\int_{D_T} f(x,t;Z)dx} \quad . \tag{1-26}$$

Note that $\rho(x,t|Z)$ and $f(x,t;Z)$ have the same spatial dependence and differ only by a function of time.

A related quantity, of interest in "time-late" problems, is the density in the absence of search:

$$\rho(x,t)dx = \Pr\{x \le X(t) \le x + dx\} \quad . \tag{1-27}$$

THE SEARCH EQUATIONS

General References: 10, 11, 12.

In order to derive the search equation, consider the time increment $(t,\ t + \Delta t)$. Then, in this increment, one finds that

$$f(x,t+\Delta t;Z)dx = \int \left(1-\hat{\psi}(x-\xi,t;Z)\Delta t\right) q(\xi,t,\Delta t,x-\xi) f(x-\xi,t;Z)\,d\xi\,dx. \quad (2\text{-}1)$$

When the right-hand side is Taylor expanded, one obtains:

$$f(x,t+\Delta t;Z)dx = \int\Bigg\{\left(1-\hat{\psi}(x,t;Z)\Delta t\right)\Bigg[q(\xi,t,\Delta t,x) f(x,t;Z) - \sum_i \xi_i \frac{\partial}{\partial x_i}(qf) + \frac{1}{2}\sum_{ij}\xi_i\xi_j \frac{\partial^2}{\partial x_i \partial x_j}(qf)\Bigg] + \Delta t\, q(\xi,t,\Delta t,x-\xi) f(x-\xi,t;Z)\cdot O(\xi) + O(\xi^3)\Bigg\} d\xi\,dx \quad (2\text{-}2)$$

$$= (1 - \hat{\psi}\Delta t)\Bigg\{\int d\xi\ q(\xi,t,\Delta t,x) f(x,t;Z) - \sum_i \frac{\partial}{\partial x_i}(\xi_i qf) + \sum_{ij}\frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(\xi_i\xi_j qf\right) + O(\xi^3)\Bigg]\Bigg\} \quad (2\text{-}3)$$

$$+ \Delta t \int q(\xi,t,\Delta t,x-\xi) f(x-\xi,t;Z) O(\xi)\,d\xi\,dx \quad .$$

$$f(x,t+\Delta t;Z) = (1 - \hat{\psi}\Delta t)\left[f(x,t;Z)\int q(\xi,t,\Delta t,x)\,d\xi\right.$$

$$- \sum_i \frac{\partial}{\partial x_i}\left(f\int \xi_i q(\xi,t,\Delta t,x)\,d\xi\right)$$

$$+ \sum_{ij} \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(f\int \xi_i\xi_j q(\xi,t,\Delta t,x)\,d\xi\right) \qquad (2\text{-}4)$$

$$\left. + \int O(\xi^3) q(\xi,t,\Delta t,x)\,d\xi\right] + o(\Delta t) \ .$$

Thus:

$$f(x,t+\Delta t;Z) = (1 - \hat{\psi}\Delta t)\left[f(x,t;Z) - \sum_i \frac{\partial}{\partial x_i}\left(\hat{b}_i(x,t)\Delta t + o(\Delta t)\right)\right. \qquad (2\text{-}5)$$

$$\left. + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\hat{a}_{ij}(x,t)\Delta t + o(\Delta t)\right) + o(\Delta t)\right] .$$

Equation 2-5 follows from the definition of $\hat{b}(x,t)$ and $\hat{a}(x,t)$. Expanding the right-hand side gives:

$$f(x,t+\Delta t;Z) - f(x,t;Z) = -\psi f\Delta t - \sum_i \frac{\partial}{\partial x_i}(\hat{b}_i f)\Delta t$$

$$+ \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i \partial x_j}(\hat{a}_{ij} f)\Delta t + o(\Delta t) \ .$$

Dividing by Δt and taking the limit as $\Delta t \to 0$ gives the search equation:

$$\frac{\partial f}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (\hat{a}_{ij} f) - \sum_i \frac{\partial}{\partial x_i} (\hat{b}_i f) - \hat{\psi} f \quad . \tag{2-7}$$

Equation 2-7 will be called the stochastic search equation (SSE).

Exercise

Suppose that there is no search. Let $\rho(x,t)dx = \Pr\{x \le X(t) \le x + dx\}$. Show that $\rho(x,t)$ satisfies the following equation:

$$\frac{\partial \rho}{\partial t} = \sum_{i,j} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (\hat{a}_{ij} f) - \sum_i \frac{\partial}{\partial x_i} (\hat{b}_i f) \quad . \tag{2-8}$$

Exercise

For the case of deterministic target motion, show that $f(x,t;Z)$ satisfies the deterministic search equation (DSE):

$$\frac{\partial f}{\partial t} = - \sum_i \frac{\partial}{\partial x_i} (\hat{b}_i f) - \hat{\psi} f \quad . \tag{2-9}$$

Remark

Define a flux $J_i(x,t;f)$ by:

$$J_i(x,t;f) = -\left[\sum_j \frac{\partial}{\partial x_j}\left(\hat{a}_{ij} f\right) - \hat{b}_i f\right] . \qquad (2\text{-}10)$$

Then, the SSE (equation 2-7) becomes:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[J_i(x,t;f)\right] = -\hat{\psi} f . \qquad (2\text{-}11)$$

This is a conservation or continuity equation.

Now consider equation 2-7. It needs boundary and initial conditions.

1. Initial Condition: $f(x,t;Z) \to \rho_0(x)$ as $t \downarrow 0$, where $\rho_0(x)$ is the initial density.

2. Boundary Conditions are more difficult. Possible ones are:

   a. $f(x,t;Z) = 0$ for x on the boundary of D_T (i.e., an absorbing boundary).

   b. The normal derivative $\partial f/\partial n = 0$ on the boundary (reflecting boundary).

In this paper, the boundaries are simply ignored. This leads to solutions that are valid far away from the boundary.

They can be modified to satisfy any needed boundary conditions (see reference 13 or 1).

Remark

$f(x,t;Z)$ does not integrate to 1.

In fact,

$$\int_{D_T} f(x,t;Z)dx = \text{Prob}\{\text{search up to time } t \text{ is not successful}\} < 1 \text{ (hopefully)}.$$

THE SCALED SEARCH EQUATION

Let T_c, L_c, b_m, a_m be defined as follows:

T_c = a characteristic time, e.g., time available for search

L_c = a characteristic distance, e.g., distance from center of $\rho_0(x)$ to the boundary

a_m = maximum value of $\hat{a}_{ij}(x,t)$ over all i,j

b_m = maximum value of $\hat{b}_i(x,t)$ over all i,j.

Now define dimensionless variables by:

$$\tau = \frac{t}{T_c} \qquad y_i = \frac{x_i}{L_c} \qquad a_{ij} = \frac{\hat{a}_{ij}}{a_m} \qquad b_i = \frac{\hat{b}_i}{b_m}$$

The search equation 2-7 takes the form:

$$\frac{1}{T_c}\frac{\partial f}{\partial \tau} = \frac{a_m}{L_c^2}\sum_{i,j}\frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}f) - \frac{b_m}{L_c}\sum_i \frac{\partial}{\partial y_i}(\bar{b}_i f) - \hat{\psi} f \ , \qquad (2\text{-}12)$$

or

$$\frac{\partial f}{\partial \tau} = \frac{a_m T_c}{L_c^2}\sum_{i,j}\frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}f) - \frac{b_m T_c}{L_c}\sum_i \frac{\partial}{\partial y_i}(\bar{b}_i f) - \hat{\psi} T_c f. \qquad (2\text{-}13)$$

Now assume that T_c, L_c are chosen so that:

$$\frac{a_m T_c}{L_c^2} \equiv \varepsilon << 1 \ . \qquad (2\text{-}14)$$

Define:

$$\left.\begin{aligned} b_i &= \bar{b}_i\left(\frac{b_m T_c}{L_c}\right) \\ \psi &= \hat{\psi} T_c \ . \end{aligned}\right\} \qquad (2\text{-}15)$$

The final non-dimensional search equations is (reverting to x,t instead of y,τ):

$$\frac{\partial f}{\partial t} = \varepsilon \sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}f) - \sum_i \frac{\partial}{\partial x_i}(b_i f) - \psi f \ . \qquad (2\text{-}16)$$

Remark

When $\varepsilon \to 0$, equation 2-16 becomes the deterministic search equation.

Remark

It has been assumed that b_i and ψ are quantities that are $0(1)$, i.e., of the order of 1. It could be that b_i are $0(\varepsilon)$, in which case equation 2-16 would take the form:

$$\frac{\partial f}{\partial t} = \varepsilon\left[\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}f) - \sum_i \frac{\partial}{\partial x_i}(b_i f)\right] - \psi f \ . \qquad (2\text{-}17)$$

For purposes here, equation 2-16 will be used instead of equation 2-17.

Remark: An Alternative Formulation for ε

Suppose that the initial data were:

$$\rho_0 = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x_1^2+x_2^2)}{2\sigma^2}\right] .$$

Under the scaling

$$x_i = y_i L_c \ ,$$

one obtains:

$$\rho_0 = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(y_1^2+y_2^2)L_c^2}{2\sigma^2}\right] .$$

If $\sigma^2/2L_c^2 << 1$, then an alternative definition of ε could be:

$$\varepsilon = \frac{\sigma^2}{2L_c^2} \quad .$$

In this case, (a_{ij}) and $b(y,t)$ would have different interpretations.

Remark

One could imagine that $b(x,t)$ in equation 2-16 is not known with complete certainty, but that $b(x,t)$ is known in a conditionally deterministic fashion. In this case, one solves an equation of the form of equation 2-16 for each possible $b(x,t)$ and then averages over α .

SOLUTION OF THE DETERMINISTIC AND CONDITIONALLY DETERMINISTIC SEARCH EQUATIONS

References: 14,15.

The deterministic search equation is:

$$\frac{\partial f}{\partial t} = - \sum_{i=1} \frac{\partial}{\partial x_i} (b_i f) - \psi f \quad , \tag{3-1}$$

with initial data:

$$f(x,0;\ Z) = \rho_0(x), \tag{3-2}$$

and no boundary data.

In order to solve equation 3-1, the method of characteristics is used. Rewrite equation 3-1 as:

$$\frac{\partial f}{\partial t} + \Sigma\, b_i \frac{\partial f}{\partial x_i} = -\left(\psi + \Sigma \frac{\partial b_i}{\partial x_i}\right) f \quad . \tag{3-3}$$

Introduce a new variable S so that:

$$\frac{dt}{dS} = 1 \quad , \tag{3-4}$$

and set:

$$\frac{dx_i}{dS} = b_i\Big(x,\ t(S)\Big). \tag{3-5}$$

Choose initial conditions so that:

$$\left.\begin{aligned} t &= 0 \qquad \text{when} \quad S = 0 \\ x_i &= x_{i0} \qquad \text{when} \quad S = 0 \end{aligned}\right\} . \tag{3-6}$$

The solutions of equations 3-4 and 3-6 give a "curve" (or space time ray) in D_T. From equation 3-3:

$$\frac{df}{dS} = \frac{\partial f}{\partial t}\frac{dt}{dS} + \sum \frac{\partial f}{\partial x_i}\frac{dx_i}{dS} \tag{3-7}$$

$$= \frac{\partial f}{\partial t} + \sum b_i \frac{\partial f}{\partial x_i} \quad . \tag{3-8}$$

Thus, on the solution curves of equations 3-4 and 3-5,

$$\frac{df}{dS} = -\left[\psi\Big(x(S),\ S,\ Z(S)\Big) + \sum \frac{\partial b_i}{\partial x_i}\Big(x(S),S\Big)\right] f \tag{3-9}$$

with

$$f = \rho_0(x_0) \qquad \text{when} \quad S = 0 \tag{3-10}$$

(see figure 2).

Formally, one can write:

$$f(x,t;Z) = \rho_0(x_0)\exp\left[-\int_0^t \left(\psi + \sum \frac{\partial b_i}{\partial x_i}\right) dS\right] \quad . \tag{3-11}$$

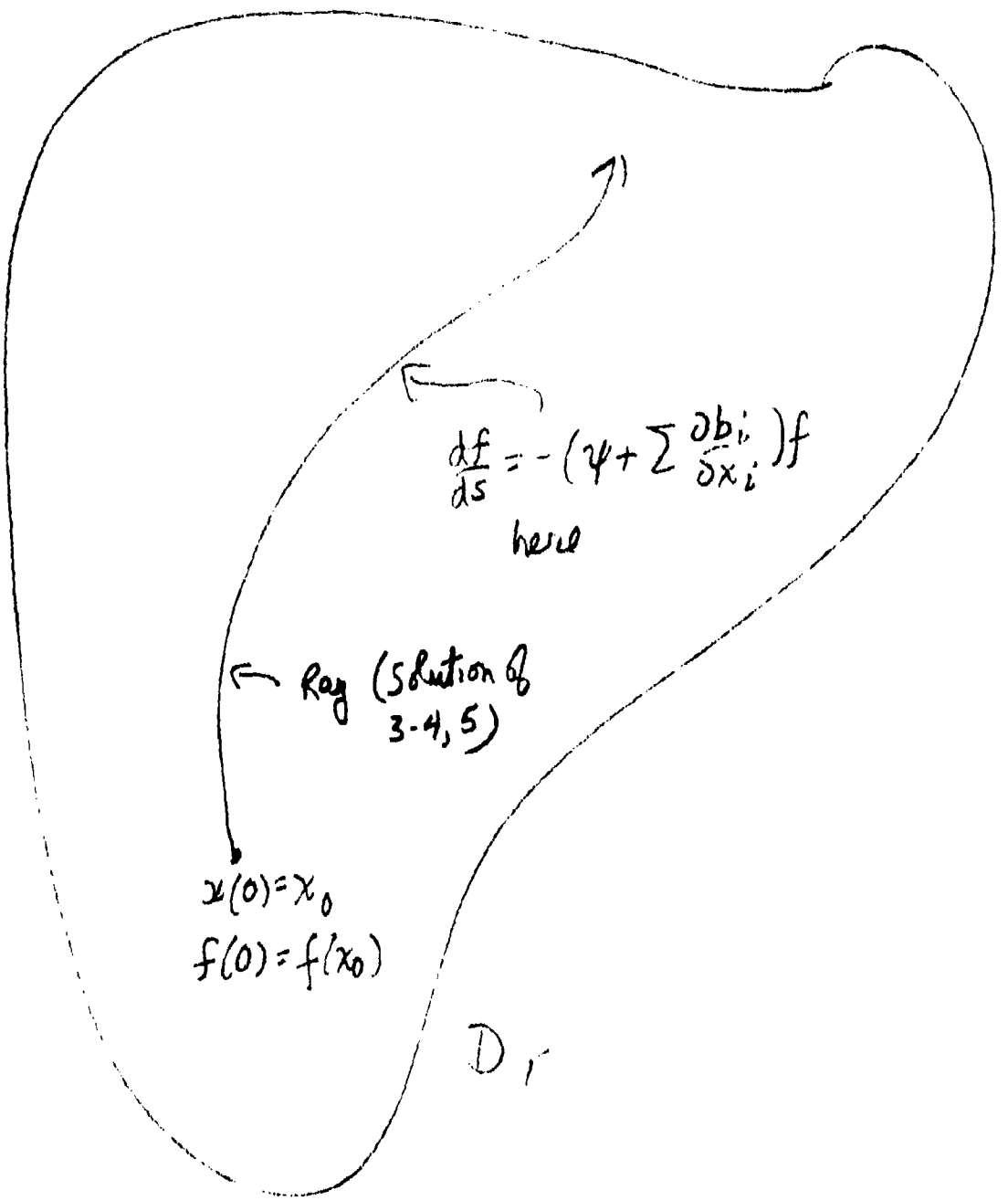

FIG. 2: SOLUTION OF THE DETERMINISTIC SEARCH EQUATION

In using this formula, one writes:

$$x_i(t) = x_{i0} + \int_0^t b_i(x,S)dS \tag{3-12}$$

or

$$x_{i0} = x_i(t) - \int_0^t b_i(x,S)dS \quad . \tag{3-13}$$

This form is especially useful when b is independent of x. In that case:

$$x_0 = x - \int_0^t b(S)dS \quad . \tag{3-14}$$

EXERCISE (A CASE WHERE EQUATION 3-13 IS USEFUL)

Suppose that $b_i = v_i(t)$, independent of x. Show that:

$$f(x,t;Z) = \rho_0\left[x - \int_0^t v(S)dS\right] \exp\left\{-\int_0^t \psi\left[x - \int_S^t v(S')dS',\ S,\ Z\right]dS\right\} \tag{3-15}$$

Write down the integral that gives the probability of detection by time t.

Remark

Note that equations 3-11 and 3-14 can be summarized as:

$f \sim$ [term due to target motion]x[term due to search] .

COMPUTATIONAL ALGORITHM FOR DETERMINISTIC TARGET MOTION

Step 1

Pick $x_0 = (x_{10}, x_{20}, x_{30})$

Step 2

Solve $\frac{dx_i}{dt} = b_i(x,t)$; $x_i(0) = x_{i0}$

Step 3

Solve $\frac{df}{dt} = -\left(\psi + \sum_i \frac{\partial b_i}{\partial x_i}\right) f$

$f(0) = \rho_0(x_0)$.

Step 4

Cycle (i.e., pick a different x_0 and return to Step 1).

COMPUTATIONAL ALGORITHM FOR CONDITIONALLY DETERMINISTIC TARGET MOTION

In this case

$$b(x,t) = b^{\alpha}(x,t) \tag{3-16}$$

with probability p_α. Let $f^{\alpha}(x,t;Z)$ satisfy:

$$\frac{\partial f^{\alpha}}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(b_i^{\alpha} f^{\alpha}) - \psi f^{\alpha} \tag{3-17}$$

$$f^{\alpha}(x,0;Z) = \rho_0(x). \tag{3-18}$$

In order to determine $f(x,t;z) = E_\alpha\{f^\alpha(x,t;z)\}$, the following computational algorithm can be used.

Step 1

Fix $b^\alpha(x,t)$ (with probability p_α).

Step 2

Pick $x_0 = (x_{10}, x_{20}, x_{30})$

Step 3

Solve: $\frac{dx_i}{dt} = b_i^\alpha(x,t)$; $x_i(0) = x_{i0}$.

Step 4

Solve: $\frac{df^\alpha}{dt} = -\left(\psi + \sum_i \frac{\partial b_i^\alpha}{\partial x_i}\right) f^\alpha \qquad f^\alpha(0) = \rho_0(x_0)$.

Step 5

Cycle to step 2 as desired.

Step 6

Cycle through $b^\alpha(x,t)$.

Step 7

Construct: $f(x,t;z) = E_\alpha\{f^\alpha(x,t;z)\}$.

ALTERNATIVE COMPUTATIONAL ALGORITHM

Step 1

Fix: $x_0 = (x_{10}, x_{20}, x_{30})$

Step 2

Fix: $b^{\alpha}(x,t)$

Step 3

Solve: $\frac{dx_i}{dt} = b_i^{\alpha}(x,t) \qquad x_i(0) = x_{i0}$

Step 4

Solve: $\frac{df^{\alpha}}{dt} = -\left(\psi + \sum \frac{\partial b_i^{\alpha}}{\partial x_i}\right) f^{\alpha}$

Step 5

Cycle through $b^{\alpha}(x,t)$

Step 6

Construct: $f(x,t;Z) = E_{\alpha}\{f(x,t;Z)\}$

Step 7

Cycle through x_0 .

SOLUTION OF THE STOCHASTIC SEARCH EQUATION IN SOME SPECIAL CASES

General References: 6, 13, 15, 16, 17, 18.

In order to motivate the solutions constructed in the next section, certain special cases are studied in this section. Study of such canonical problems (see reference 16) has been very useful in other disciplines.

First consider the stochastic search equation:

$$\frac{\partial f}{\partial t} = \sum \frac{\varepsilon}{2} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} f) - \sum \frac{\partial}{\partial x_i} (b_i f) - \psi f \quad , \tag{4-1}$$

with the following assumptions:

A1) $a_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

A2) $b_i(x,t) = b_i$ a constant independent of position and time.

A3) The target moves in the plane.

A4) $\psi(x,t;Z) = \bar{\psi}$, a constant.

The search equation is then:

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \left[\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \right] - \left[b_1 \frac{\partial f}{\partial x_1} + b_2 \frac{\partial f}{\partial x_2} \right] - \bar{\psi} f \quad , \tag{4-2}$$

with

$$f(x,0;Z) = \rho_0(x) \tag{4-3}$$

Exercise

Show that if one sets $f(x,t;Z) = w(x,t)e^{-\bar{\psi}t}$, then:

$$\frac{\partial w}{\partial t} = \frac{\varepsilon}{2}\left[\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right] - \left[b_1 \frac{\partial w}{\partial x_1} + b_2 \frac{\partial w}{\partial x_2}\right] \tag{4-4}$$

with $w(x,0) = \rho_0(x)$.

CANONICAL PROBLEM: TYPE 1 INITIAL DATA

Assume that:

$$\rho_0(x) = \delta(x) \tag{4-5}$$

i.e., $\Pr\{X(0) = 0\} = 1$.

Then,

$$w(x,t) = \frac{1}{2\pi\varepsilon t} \exp\left\{-\left[\frac{(x_1-b_1t)^2+(x_2-b_2t)^2}{2\varepsilon t}\right]\right\} . \tag{4-6}$$

The right-hand side of equation 4-6 is the Green's function, $G(x,t)$, or fundamental solution. As $t \downarrow 0$

$$w(x,t) \to \delta(x) = \delta(x_1)\delta(x_2) . \tag{4-7}$$

Exercise

Show that equation 4-6 satisfies equation 4-4.

CANONICAL PROBLEM: GENERAL INITIAL DATA

Assume that:

$$\rho_0(x) = \bar{\rho}(x_1, x_2) \quad .$$

Then (see references 5, 14, or 15)

$$w(x,t) = G * \rho$$

$$= \int G(x - \xi, t)\rho_0(\xi)d\xi \quad . \tag{4-8}$$

Exercise

Assume that:

$$\rho_0(x) = \frac{1}{2\pi\sigma^2} \exp\left(- \frac{x_1^2+x_2^2}{2\sigma^2}\right) \quad .$$

Find $w(x,t)$.

Hint: Gaussian functions are closed under convolution (complete the square).

CANONICAL PROBLEM: TYPE 3 INITIAL DATA

Assume that:

$$\rho_0(x) = \begin{cases} \bar{\rho}(x) & \text{in square of side } 2\ell \\ 0 & \text{outside} \end{cases} \tag{4-9}$$

Then:

$$w(x,t) = \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} G(x-\xi,\ t)\bar{\rho}(\xi)d\xi \tag{4-10}$$

Remark

The one-dimensional analoque of equation 4-9 is:

$$\rho_0(x) = \begin{cases} \bar{\rho}(x) & -\ell \le x \le \ell \\ 0 & \text{otherwise}, \end{cases} \tag{4-11}$$

and of equation 4-11, is

$$w(x,t) = \frac{1}{\sqrt{2\pi\varepsilon t}} \int_{-\ell}^{\ell} \bar{\rho}(\xi)\exp\left[-\frac{(x-bt-\xi)^2}{2\varepsilon t}\right]d\xi\ . \tag{4-12}$$

(Hard) Exercise (see references 17 and 18)

Use integration by parts to develop an asymptotic expansion of equation 4-12 valid for small ε and $|(x-bt)| < \ell$.

Remark

All of the above results also hold when $b_i(x,t) = b_i(t)$ only, independent of x. In this case, the Green's function $G(x,t)$ is:

$$G(x,t) = \frac{1}{2\pi\varepsilon t} \exp\left\{ - \frac{1}{2\varepsilon t}\left[\left(x_1 - \int_0^t b_1(S)dS\right)^2 + \left(x_2 - \int_0^t b_2(S)dS\right)^2\right]\right\}. \tag{4-13}$$

CANONICAL PROBLEM: NONCONSTANT ψ

Allow ψ to vary, but keep all the other assumptions. The search equation can be rewritten as:

$$\frac{\partial f}{\partial t} - \frac{\varepsilon}{2}\left[\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right] + \left[b_1 \frac{\partial f}{\partial x_1} + b_2 \frac{\partial f}{\partial x_2}\right] = -\psi f \equiv u(x,t) . \tag{4-14}$$

The solution of equation 4-14 is formally

$$f(x,t;Z) = G * u . \tag{4-15}$$

But, $u = -\psi f$, so that:

$$\begin{aligned} f(x,t;Z) &= G^*(-\psi f) \\ &= -G^*(\psi^*(G^*u)) \\ &\quad \text{etc.} \end{aligned} \tag{4-16}$$

The iterative procedure of equation 4-16 and a suitable fixed point theorm can probably be used to prove the existence of solutions to equation 4-14.

CANONICAL PROBLEM: ORNSTEIN-UHLENBECK PROCESS

Assume that the target moves in one dimension with:

$$\left.\begin{aligned} b(x,t) &= -\beta x \\ a(x) &= 1 \end{aligned}\right\} \tag{4-17}$$

and $\psi = \bar{\psi}$, a constant. Then, if $f(x,t;Z) = w(x,t)e^{-\bar{\psi}t}$, $w(x,t)$ satisfies:

$$\frac{\partial w}{\partial t} = \frac{\varepsilon}{2}\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial}{\partial x}(wx) \ . \tag{4-18}$$

Assume that $w(x,0) = \delta(x-x_0)$. Feller (reference 5) has shown that:

$$w(x,t) = \left[\frac{\beta}{\pi\varepsilon(1-e^{-2\beta t})}\right]^{\frac{1}{2}} \exp\left[\frac{-\beta\left(x-x_0e^{-2\beta t}\right)^2}{\varepsilon(1-e^{-2\beta t})}\right] \tag{4-19}$$

Remark

All of these problems have solutions of the form:

$$f \sim w(x,t)e^{-\psi t} = [\text{term due to target motion}] \times [\text{term due to search}] \ .$$

APPROXIMATE SOLUTION OF THE STOCHASTIC SEARCH EQUATION IN THE GENERAL CASE

General references: 1, 6, 10, 13, 14, 16, 17.

Reconsider the stochastic search equation:

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \sum \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} f) - \sum \frac{\partial}{\partial x_i} (b_i f) - \psi f \quad , \tag{5-1}$$

with initial data:

$$f(x,0;Z) = \rho_0(x) \quad . \tag{5-2}$$

For simplicity, assume that the initial data is exponential:

$$\rho_0(x) = e^{-\Phi(x)/\varepsilon} \sum_{k=0} \varepsilon^k h_k(x) \tag{5-3}$$

Exercise

Cohen and Lewis (reference 13) have shown that the following "trick" can be used to obtain data in the form of equation 5-3. Show that formally one can set:

$$\begin{aligned} \Phi(x) &= -\varepsilon \ell n \rho_0(x) \\ h_0(x) &= 1 \\ h_k(x) &\equiv 0 \qquad k \geq 1 \quad . \end{aligned} \tag{5-4}$$

For

$$\rho_0(x) = \frac{1}{2\pi\sigma^2} e^{-(x_1^2+x_2^2)/2\sigma^2} \tag{5-5}$$

find $\phi(x)$ and the $h_k(x)$.

Hint: Redefine σ, in terms of ε.

Based on the solutions of the canonical problems, one seeks a solution of the SSE in the form (note ϕ does not depend on $Z(t)$):

$$f(x,t;Z) = e^{-\phi(x,t)/\varepsilon} \sum_{k=0} \varepsilon^k g_k(x,t;Z) \quad . \tag{5-6}$$

In this equation, $\phi(x,t)$ and $g_k(x,t;Z)$ $k = 0,1,2,\ldots$ are to be determined. Equation 5-6 is called a "ray ansatz."

Now apply the following procedure:

1. Evaluate $\partial f/\partial t$, $\partial f/\partial x_i$ and $\partial^2 f/\partial x_i \partial x_j$ if $f(x,t;Z)$ is given by equation 5-6.

2. Substitute into the search equation equation 5-1.

3. Collect terms according to powers of ε. One obtains:

$$e^{-\phi(x,t)/\varepsilon}\left[\frac{1}{\varepsilon}\{\ \} + \varepsilon^0\{\ \} + \varepsilon^1\{\ \} + \ldots\right] = 0. \tag{5-7}$$

4. Set each coefficient of ε^k equal to zero. This gives an equation for $\phi(x,t)$ if $k=-1$ and for $g_k(x,t;Z)$ if $k=0,1,2\ldots$

Exercise

Show that the equation that $\phi(x,t)$ satisfies is:

$$\frac{\partial \phi}{\partial t} + \sum_i b_i \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{i,j} a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} = 0 . \tag{5-8}$$

Equation 5-8 is the "eikonal" or Hamilton-Jacobi equation. It is first order, but non-linear. Let:

$$H(x,p,t) = \sum b_i p_i + \frac{1}{2} \sum a_{ij} p_i p_j , \tag{5-9}$$

be a "Hamiltonian" in which $p = (p_1, p_2, p_3)$ is a new independent variable. Then one writes:

$$\frac{\partial \phi}{\partial t} + H\left(x, \frac{\partial \phi}{\partial x}, t\right) = 0 . \tag{5-10}$$

Equation (5-8) can also be solved by the method of characteristics. (references 14, 15, and 16)

Theorem (reference 14,15)

Let $x(t)$ and $p(t)$ be the solutions of:

$$\left.\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i} , \qquad x_i(0) = x_{i0} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i} , \qquad p_i(0) = p_{i0} \end{aligned}\right\} . \tag{5-11}$$

These are called the ray equations. On the solution curves of equation 5-11:

$$\frac{d\phi}{dt} = -H(x,p,t) + \sum p_i \frac{dx_i}{dt} = \frac{1}{2} a_{ij} p_i p_j \qquad (5\text{-}12)$$

$$\phi(0) = \phi_0$$

Exercise

1) Assume that:

$$(a_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $b_i(x,t) = b_i$, a constant. Write down and solve the ray equations.

2) Assume that

$$(a_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $b_i(x,t) = b_i(t)$ only. Write down the ray equations. When can they be solved?

3) What are the ray equations for general (a_{ij}) and $b_i(x,t)$?

Theorem (references 14 and 15)

If the solution curves of equation 5-11 don't intersect, then $p_i = \partial\phi/\partial x_i$ and $\phi(x,t)$, the solution of equation 5-12, is also a solution of equation 5-8.

Initial conditions are still needed. In light of equations 5-3 and 5-6, one has:

$$e^{-\phi(x)/\varepsilon} \sum_k \varepsilon^k h_k(x) = e^{-\phi(x,0)/\varepsilon} \sum q_k(x,0,z)\varepsilon^k \quad . \tag{5-13}$$

Hence, one sets:

$$\begin{aligned} x(0) &= x_0 \\ p_i(0) &= \left.\frac{\partial \phi}{\partial x_i}\right|_{x_0} \\ \phi(x_0,0) &= \phi(x_0) \end{aligned} \tag{5-14}$$

Thus, the following algorithm can be used.

ALGORITHM FOR SOLUTION OF THE HAMILTON-JACOBI EQUATION

1. Pick x_0 (arbitrary).

2. Solve:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i} & x_i(0) &= x_{i0} \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i} & p_i(0) &= \left.\frac{\partial \phi}{\partial x_i}\right|_{x_0} \end{aligned} \tag{5-15}$$

3. Set $\phi(x,0) = \phi_0 = \phi(x_0)$ and solve:

$$\frac{d\phi}{dt} = \frac{1}{2} \sum a_{ij} p_i p_j \quad . \tag{5-16}$$

4. Cycle through x_0 .

ALTERNATIVE ALGORITHM FOR THE SOLUTION OF THE HAMILTON-JACOBI EQUATION

The following alternative algorithm can also be used.

1. Choose a point x at time t.

2. Find the point x_0 such that the solution of

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad x_i(0) = x_{i0}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad p_i(0) = \left.\frac{\partial \phi}{\partial x_i}\right|_{x_0}$$

passes through the point x at time t.

3. Set $\phi_0 = \phi(x_0)$ and solve

$$\frac{d\phi}{dt} = \frac{1}{2} \sum a_{ij} p_i p_j \, .$$

This generates $\phi(x,t)$.

4. Cycle through x at time t.

Variational Interpretation (references 6, 10, 14, 16)

Define a Lagrangian $L(x,dx/dt)$ by:

$$L\left(x,\frac{dx}{dt},t\right) + H(x,p,t) = \sum_i \frac{dx_i}{dt} p_i \quad . \tag{5-17}$$

Exercise

For $H(x,p)$ given by equation 5-9, show that:

$$L\left(x, \frac{dx}{dt}, t\right) = \frac{1}{2} \sum_{i,j} \left(\frac{dx_i}{dt} - b_i\right)(a_{ij})^{-1}\left(\frac{dx_j}{dt} - b_j\right) \quad . \tag{5-18}$$

In this equation, $(a_{ij})^{-1}$ is the inverse of (a_{ij}):

$$(a_{ij})(a_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

Let

$$C\{t,x\} = \{\text{set of all paths } u(S) \text{ with } u(0)=x_0 \text{ and } u(t)=x\}$$

(see figure 3). Then, according to Hamilton's principle:

$$\phi(x,t) = \min_{C\{t,x\}} \int_0^t L\left(u, \frac{du}{dS}, S\right) dS \quad . \tag{5-19}$$

This formulation is often very useful for numerical evaluation of $\phi(x,t)$.

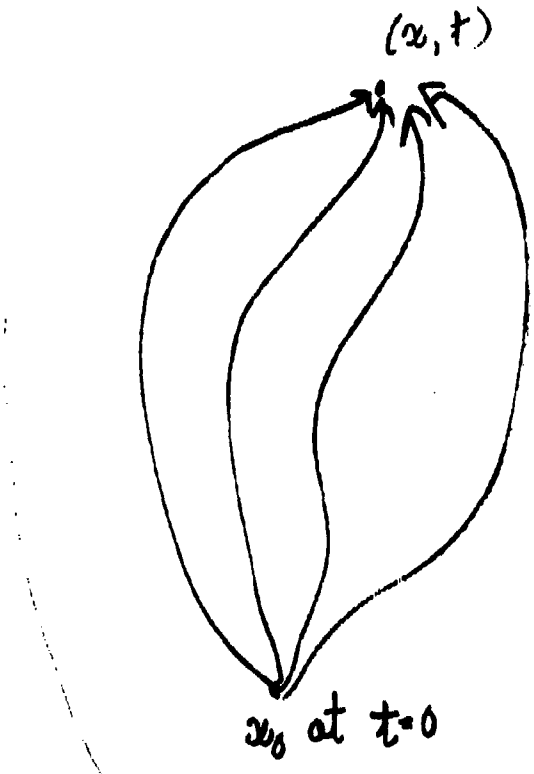

FIG. 3: ILLUSTRATING HAMILTON'S PRINCIPLE

Remarks

1. If there is a deterministic trajectory connecting $(x_0,0)$ and (x,t), then $L\left(x, \frac{dx}{dt}, t\right)$ vanishes on that trajectory.
2. If $\rho_0(x) = \delta(x-x_0)$ then the p_i's can be treated as parameters (see reference 1 or 6).
3. If $\rho_0(x)$ is type 3, i.e.:

$$\rho_0(x) = \begin{cases} \bar{\rho}(x) & x \in D_T' \subset D_T \\ 0 & x \notin D_T' \end{cases} .$$

there may be difficulties with the ansatz (equation 5-6) (i.e., rays may intersect for x_0 near the boundary of D_T'). More work (boundary layer solutions or uniform methods) can eradicate the difficulties.

Now consider $g_0(x,t;Z)$.

Exercise

Show that $g_0(x,t;Z)$ satisfies the following equation:

$$\frac{\partial g_0}{\partial t} + \sum_i \left(\sum_j a_{ij} \frac{\partial \phi}{\partial x_j} + b_i \right) \frac{\partial g_0}{\partial x_i}$$

$$= - \left[\sum_{i,j} \left(\frac{\partial a_{ij}}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \frac{1}{2} a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) + \sum_i \frac{\partial b_i}{\partial x_i} + \psi(x,t;Z) \right] g_0 . \qquad (5\text{-}20)$$

This is a first-order linear differential equation for $g_0(x,t;z)$; it is called a transport equation. Note that if $(a_{ij}) = (0)$, then equation 5-20 becomes the deterministic search equation.

To solve equation 5-20, recall that:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = b_i + \sum_j a_{ij} \frac{\partial \phi}{\partial x_j} \quad . \tag{5-21}$$

Then equation 5-20 becomes:

$$\frac{\partial g_0}{\partial t} + \sum \frac{dx_i}{dt} \frac{\partial g_0}{\partial x_i} = -\left[\sum_{i,j} \left(\frac{\partial a_{ij}}{\partial x_i} p_j + \frac{1}{2} a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) + \sum_i \frac{\partial b_i}{\partial x_i} + \psi(x,t;z) \right] g_0 \quad . \tag{5-22}$$

Thus, on the rays:

$$\frac{dg_0}{dt} = -\left\{ \Gamma(t) + \psi\Big(x(t),\ t,\ z\Big) \right\} g_0 \quad . \tag{5-23}$$

In this equation:

$$\Gamma(t) = \sum_{i,j} \left(\frac{\partial a_{ij}}{\partial x_j} p_j + \frac{1}{2} a_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) + \sum_i \frac{\partial b_i}{\partial x_i} \quad . \tag{5-24}$$

An alternative formulation is obtained by following reference 6. Let $J(t)$ satisfy:

$$\frac{dJ}{dt} = J \sum_i \frac{\partial}{\partial x_i} \left(\frac{dx_i}{dt} \right) . \tag{5-25}$$

Then equation 5-24 becomes:

$$\Gamma(t) = \frac{1}{2J} \frac{dJ}{dt} + \frac{1}{2} \sum_{i,j} \frac{\partial a_{ij}}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \sum_i \frac{\partial b_i}{\partial x_i} , \tag{5-26}$$

so that $\partial^2 \phi / \partial x_i \partial x_j$ does not have to be computed.

Remark

Ludwig (reference 6) interprets J as the Jacobian of the transformation from physical to "ray space."

Initial Condition

From equation 5-3 and 5-6, it is clear the the appropriate initial data is:

$$g_0(x_0, 0; z) = h_0(x_0) . \tag{5-27}$$

The other functions $g_k(x,t;z)$ are determined in a similar fashion.

SUMMARY OF COMPUTATIONAL ALGORITHMS

In this section, three algorithms are given for the solution of the three search equations corresponding to different types of target motion.

DETERMINISTIC TARGET MOTION

Equation:

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(b_i f) - \psi f$$

$$f(x,0;Z) = \rho_0(x)$$

Algorithm

1. Pick x_0

2. Solve

$$\frac{dx_i}{dt} = b_i(x,t) \qquad x_i(0) = x_{i0}$$

3. Solve

$$\frac{df}{dt} = -\left[\sum_i \frac{\partial b_i(x(t),t)}{\partial x_i} + \psi(x(t),\ t,\ Z)\right] f$$

with $f(0) = \rho_0(x_0)$

4. Cycle to 1 or 2.

CONDITIONALLY DETERMINISTIC TARGET MOTION

Equation:

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left(b_i^{\alpha} f\right) - \psi f \qquad (1)$$

$$f(x,0;Z) = \rho_0(x)$$

$b_\alpha(x,t)$ drawn from a distribution $B(b_\alpha)$.

Algorithm

1. Fix α and let $f(x,t;Z|\alpha)$ denote the solution of (1) with α fixed.

2. Obtain $f(x,t;Z|\alpha)$ by using steps 1 through 4 of the previous algorithm.

3. Calculate:

$$f(x,t;Z) = E_\alpha\{f(x,t;Z|\alpha)\} = \int_{B(b_\alpha)} f(x,t;Z|\alpha)\,dP_\alpha .$$

STOCHASTIC TARGET MOTION

Equation:

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} f) - \sum_i \frac{\partial}{\partial x_i}(b_i f) - \psi f$$

$$f(x,0;Z) = e^{-\Phi(x)/\varepsilon}\sum_{k=0}\varepsilon^k h_k(x) \quad .$$

Algorithm for $f(x,t;z) \sim e^{-\phi(x,t)/\varepsilon} g_0(x,t;z)$

1. Pick x at t ; find x_0.

2. Find $p_{i0} = \left.\frac{\partial \phi}{\partial x_i}\right|_{x=x_0}$

3. Solve the ray equations:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad x_i(0) = x_{i0}$$

$$\frac{\partial p_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad p_i(0) = p_{i0}$$

where

$$H(x,p,t) = \sum b_i p_i + \frac{1}{2} \sum a_{ij} p_i p_j \ .$$

4. Solve for ϕ :

$$\frac{d\phi}{dt} = \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j$$

$$\phi(0) = \phi(x_0)$$

5. Evaluate the Jacobian $J(t)$ or calculate $\partial^2 \phi / \partial x_i \partial x_j$

6. Solve for g_0:

$$\frac{dg_0}{dt} = -\left[\Gamma(t) + \psi(x,t,z)\right] g_0$$

$$g_0(x_0,0,z) = h_0(x_0)$$

7. Cycle to 1.

OPTIMAL SEARCH FOR A MOVING OBJECT

General References: 2, 3, 17.

In this section, the problem of optimal search is formulated and a method of solution is sketched.

Let $C\{Z\}$ be the set of allowable search paths. For example:

$$C\{Z\} = \{Z(t): Z(0)=z_0,\ Z(T_f)=z_f, \left|\frac{dz}{dt}\right| \leq v,\ \left|\frac{d^2z}{dt^2}\right| \leq a\}\ , \tag{7-1}$$

is the set of search paths with constrained endpoints and constraints on velocity and acceleration.

Three possible objective functionals are the following. (Other functionals could be used.)

FUNCTIONAL #1: INSTANTANEOUS PROBABILITY OF DETECTION

Let:

$$J_1\{Z,t\} = E_x\{\psi(x,t,Z)\} \tag{7-2}$$

$$= \int \psi(x,t,Z)\rho(x,t|Z)dx\ . \tag{7-3}$$

Note that $J_1\{Z,t\}$ is the (averaged) instantaneous probability of detection. A search plan that maximizes $J_1\{Z,t\}$ is called <u>myopic</u>.

From the definition of $\rho(x,t|Z)$, one obtains that:

$$J_1\{Z,t\} = \frac{\int \psi(x,t,Z)f(x,t;Z)dx}{\int f(x,t;Z)dx}\ . \tag{7-4}$$

When $J_1\{Z,t\}$ is extremized, one has the following interpretation. Consider an interval $(0,T)$. Divide it into k subintervals (t_i, t_{i+1}) with $t_0 = 0$ and $t_k = T$.

Next, maximize $J_1\{Z,t_i\}$ for $i = 1,2,\ldots,k-1$. The optimal path is the piecewise linear set $[Z_0, Z_1]$, $[Z_1, Z_2]$, etc., where

$$Z_i = Z(t_i).$$

As $k \to \infty$, a continuous curve $Z^*(t)$ is obtained.

FUNCTIONAL #2: PROBABILITY OF DETECTION BY TIME T_S

Let

$$J_2\{Z,T_S\} = 1 - \int f(x,T_S;Z)dx \quad . \tag{7-5}$$

This functional, $J_2\{Z,T_S\}$, is the probability of detection by time T_S. One seeks the path $Z^*(\tau)$, $0 \le \tau \le T_S$ that maximizes $J_2\{Z,T_S\}$. Also, note that $f(x,t;Z)$ satisfies:

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} f) - \sum_i \frac{\partial}{\partial x_i} (b_i f) - \psi f \quad . \tag{7-6}$$

In terms of control theory, this is an optimal control problem for a distributed parameter system with control in the coefficients. There is essentially no theory for this sort of problem.

FUNCTIONAL # 3: MEAN TIME TO DETECTION

The probability density for the time of detection can be determined as follows. Let T_F be the total time available for search. Let $h(t)dt = \Pr\{\text{detection occurs in } (t, t+dt)\}$. Then:

$$h(t) = \begin{cases} -\frac{\partial}{\partial t}\int f(x,t;Z)dx & t \le T_F \\ 0 & t > T_F \end{cases} \quad , \tag{7-7}$$

is the desired probability density.

The mean time to detection is then:

$$J_3\{Z\} = \int_0^{T_F} t\,h(t)dt \tag{7-8}$$

$$= -\int_0^{T_F} t\left[\frac{\partial}{\partial t}\int f(x,t;Z)dx\right]dt \quad . \tag{7-9}$$

SOLUTION OF AN OPTIMAL SEARCH PROBLEM

Consider now the problem of finding $Z^*(\tau) \in C\{Z\}$ that maximizes $J_2\{Z,T_S\}$. In order to make any progress, the functional will be converted to a function as follows. The functional $J_2\{Z,t\}$ can be rewritten as:

$$J_2(Z;t) = 1 - \int e^{-\phi(x,t)/\varepsilon}\sum \varepsilon^k g_k(x,t;Z)dx$$

$$= 1 - \sum \varepsilon^k \int e^{-\phi(x,t)/\varepsilon} g_k(x,t;Z)dx \tag{7-10}$$

$$= 1 - \sum_{k=0} \varepsilon^k I_k(Z,t) \quad . \tag{7-11}$$

The integrals $I_k(Z,t)$ are defined by:

$$I_k(Z,t) = \int e^{-\phi(x,t)/\varepsilon} q_k(x,t;Z)dx \quad . \tag{7-12}$$

These integrals can be analyzed by Laplace's method (reference 17).

Let $x^*(t)$ denote the minimum over D_T of $\phi(x,t)$. For simplicity, assume that there is only one minimum. From reference 17, page 338, one has that:

$$I_k(Z,t) \sim e^{-\phi(x^*(t),t)/\varepsilon} (2\pi\varepsilon)^{n/2} \sum_{j=0}^{\infty} \frac{\Delta_\xi^j G_{k0}\Big|_{\xi=0}}{j!2^j} \varepsilon^j \quad . \tag{7-13}$$

In this equation, $n = 2$ or 3 is the dimension of the space in which the target moves. The variables ξ_i are new coordinates (reference 17, page 334),

$$\Delta_\xi^j = \left(\sum_{i=1} \frac{\partial^2}{\partial \xi_i^2}\right)^j \tag{7-14}$$

and

$$G_{k0} = \frac{q_k(x^*(t),t;Z)}{\left[\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)\Big|_{x^*(t)}\right]^{\frac{1}{2}}} \quad . \tag{7-15}$$

It is often sufficient to use only the first one or two terms of the full asymptotic expansion.

Thus, $J_2\{Z,T_S\}$ has now been converted to a function, rather than functional:

$$J_2\{Z,T_S\} = 1 - (2\pi\varepsilon)^{n/2} e^{-\phi(x^*(T_S),T_S)/\varepsilon} \sum_{k=0}^{\infty} \varepsilon^k \left[\sum_{j=0}^{\infty} \frac{\Delta_\xi^j G_{k0}}{j!} \left(\frac{\xi}{2}\right)^j \right] . \tag{7-16}$$

One now determines the function $Z^*(t)$ that maximizes equation 7-14 on $[0, T_S]$. Here, $Z(t)$ occurs in the G_{k0}-terms, which depend on $g_k(x,t;Z)$ by equation 7-15, and the dependence of g_k on $Z(t)$ is given by equations such as 5-23.

Remarks

1. It is still, in most cases, necessary to use a computer to obtain $Z^*(t)$. However, by going from functional to function, the amount of work required has been vastly reduced.
2. It may be possible to solve some of these problems by "dynamic programming."

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APPENDIX A
SOLUTIONS OF EXERCISES

In this appendix, written by James Thomas, Jr., the solutions of most of the exercises are given.

EXERCISE:

$$\delta(S) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-S^2 n/2}$$

$$= \lim_{n\to\infty} \xi_n(S)$$

$$\text{with } \xi_n(S) = \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-S^2 n/2} \quad .$$

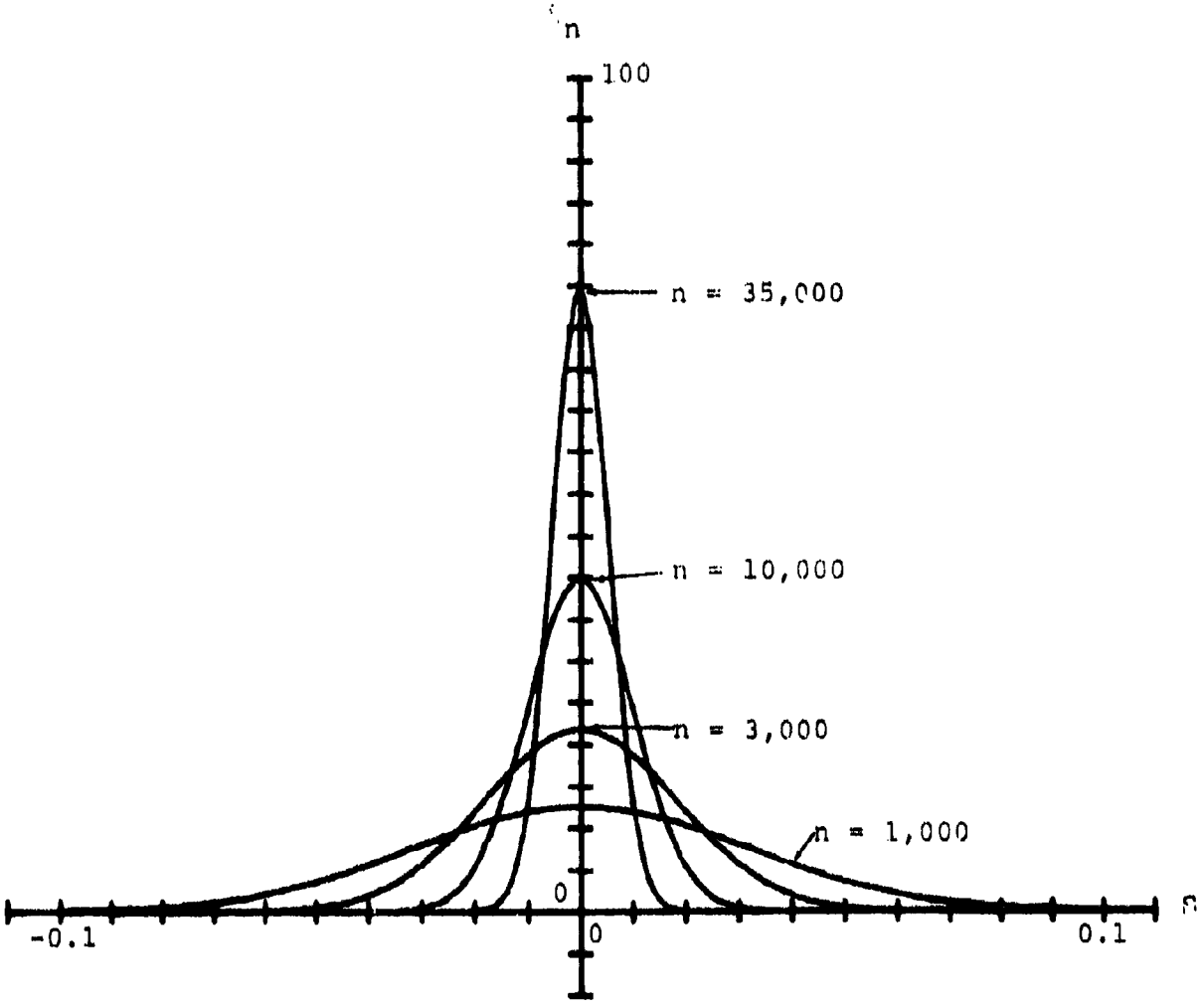
Sketch a few $\xi_n(S)$.

Sketch the corresponding $\xi_n'(S)$, where the prime indicates a derivative with respect to S .

SOLUTION:

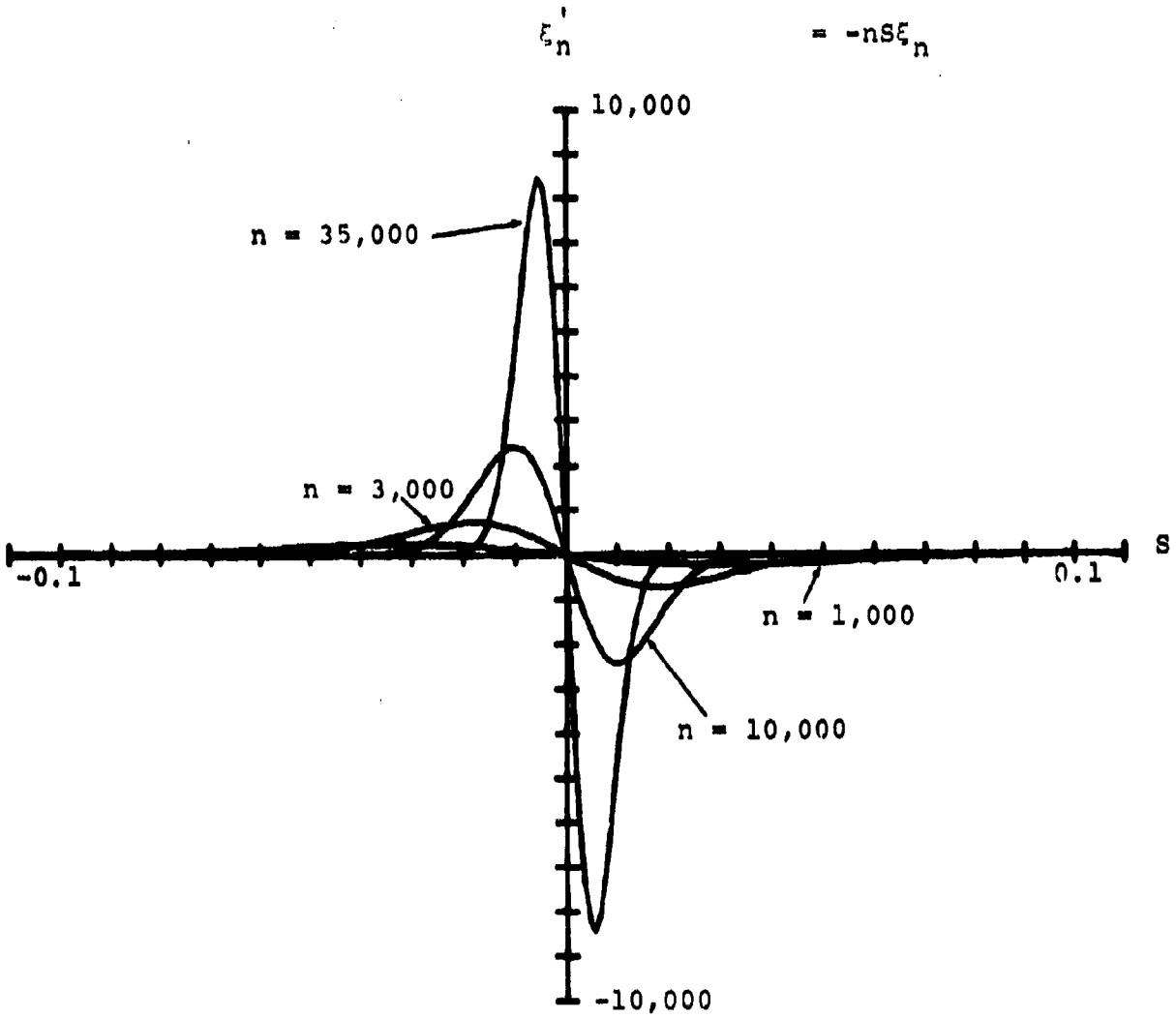
Refer to the following computer graphics.

$$f_n = \sqrt{\frac{n}{2\pi}}\, e^{-n\delta^2/2}$$



$$\xi_n' = -\frac{n^{3/2}}{\sqrt{2\pi}} Se^{-nS^2/2}$$

$$= -nS\xi_n$$



EXERCISE:

Show that for deterministic target motion:

$$\text{var}\{\Delta X | X(t)=x\} = 0.$$

SOLUTION:

For deterministic target motion, the transition function, $q(\xi,t,\Delta t,x)$ is given by

$$q(\xi,t,\Delta t,x) = \delta(\xi - b(x,t)\Delta t).$$

Substitution of this in the expression,

$$E\{(\Delta X)^2\} = \int \xi^2 q d\xi ,$$

yields

$$E\{(\Delta X)^2\} = \int \xi^2 \delta(\xi - b(x,t)\Delta t)d\xi = [b(x,t)]^2(\Delta t)^2.$$

Consequently,

$$\text{var}\{\Delta X\} = E\{(\Delta X)^2\} - [E\{\Delta X\}]^2 = 0 .$$

For stochastic target motion, given $X(t) = x$, ΔX is distributed with mean

$$b(x,t)\Delta t + o(\Delta t)$$

and covariance

$$\varepsilon a_{ij}(x,t)\Delta t + o(\Delta t) ,$$

where

$$\frac{o(\Delta t)}{\Delta t} \to 0 \qquad \text{as} \quad \Delta t \to 0 .$$

Alternatively:

$$E\{\Delta X_i \mid X(t)=x\} = b_i(x,t)\Delta t + o(\Delta t)$$

and

$$E\{\Delta X_i \Delta X_j \mid X(t)=x\} = \varepsilon a_{ij}\Delta t + o(\Delta t).$$

EXERCISE:

If ΔX is normally distributed, what happens as $\varepsilon \to 0$ with a_{ij} bounded? Write the alternative conditions for stochastic target motion in terms of q .

SOLUTION: Let $dX = X(t + dt) - X(t)$.

In several dimensions,

$$q(\xi_1, \xi_2, \ldots, \xi_i, \xi_j, t, dt, x)d\xi_1 d\xi_2 \ldots d\xi_i d\xi_j$$

$$= \text{Prob}\{\xi_1 \le X_1(t+dt) - X_1(t) \le \xi_1 + d\xi_1,\ \xi_2 \le X_2(t+dt) - X_2(t) \le \xi_2 + d\xi_2,$$

$$\ldots,\ \xi_i \le X_i(t+dt) - X_i(t) \le \xi_i + d\xi_i,\ \xi_j \le X_j(t+dt) - X_j(t) \le \xi_j + d\xi_j \mid X(t) = x\} \ .$$

If two new functions are defined such that

$$q_{ij}(\xi_i, \xi_j, t, dt, x) = \int \ldots \int d\xi_1 d\xi_2 \ldots d\xi_{i-1} q(\xi_1, \xi_2, \ldots, \xi_i, \xi_j, t, dt, x)$$

and

$$q_i(\xi_i, t, dt, x) = \int \ldots \int d\xi_1 d\xi_2 \ldots d\xi_{i-1} d\xi_j q(\xi_1, \xi_2, \ldots, \xi_i, \xi_j, t, dt, x) \ ,$$

then the alternative conditions for stochastic target motion can be written

$$E\{dX_i \mid X(t) = x\} = b_i(x,t)dt + o(dt)$$

$$= \int d\xi_i q_i(\xi, t, dt, x)\xi_i$$

and

$$E\{dX_i\ dX_j \mid X(t) = x\} = \varepsilon a_{ij} dt + o(dt)$$

$$= \iint d\xi_i d\xi_j q_{ij}(\xi_i, \xi_j, t, dt, x)\xi_i \xi_j \ .$$

Now, given that dX is normally distributed with mean,

$$b(x,t)dt + o(dt) \ ,$$

and covariance,

$$\varepsilon a_{ij}(x,t)dt + o(dt) \ ,$$

$$q_{ij}(\xi_i,\xi_j,t,dt,x) = \frac{1}{\sqrt{2\pi[\varepsilon a_{ij}dt+o(dt)]}} \cdot \exp\left\{-\frac{[\xi_i-b_i dt-o(dt)][\xi_j-b_j dt-o(dt)]}{2[\varepsilon a_{ij}dt+o(dt)]}\right\}$$

As $\varepsilon \to 0$ with a_{ij} bounded, $q_{ij}(\xi_i,\xi_j,t,dt,x)$ becomes a delta function analogous to the $\xi_n(S)$ of the first exercise. Consequently, the target motion becomes deterministic in X, i.e.,

$$dX_i = b_i(x,t)dt \qquad \text{with probability } 1 \ .$$

EXERCISE:

For one-dimensional target motion with $dX \sim N\Big(b(x,t)dt + o(dt);\ \varepsilon a(x,t)dt + o(dt)\Big)$, show that

$$\int \xi^n q(\xi,t,dt,x)d\xi = o(dt)$$

for $n \geq 3$, where

$$\frac{o(dt)}{dt} \to 0 \quad \text{as} \quad dt \to 0 \ .$$

SOLUTION:

For $dX \sim N\Big(b(x,t)dt + o(dt);\ \varepsilon a(x,t)dt + o(dt)\Big)$,

$$q(\xi,t,dt,x) = \frac{1}{\sqrt{2\pi\left[\varepsilon a(x,t)dt+o(dt)\right]}}\, e^{-\frac{\left[\xi-b(x,t)dt-o(dt)\right]^2}{2\left[\varepsilon a(x,t)dt+o(dt)\right]}} \ .$$

For convenience, let

$$q(\xi,t,dt,x) = \frac{1}{\sqrt{2\pi \cdot A}}\, e^{-\frac{\left[\xi-B\right]^2}{2A}}$$

with $A = \varepsilon a(x,t)dt+o(dt)$

and $B = b(x,t)dt+o(dt)$.

Now

$$\int \xi^n q(\xi,t,dt,x)d\xi = \frac{1}{\sqrt{2\pi A}} \int \xi^n e^{-\frac{\left[\xi-B\right]^2}{2A}}\, d\xi \ .$$

Making the change of variable,

$$y = \xi - B,$$

$$\int \xi^n q(\xi, t, dt, x)\, d\xi = \frac{1}{\sqrt{2\pi A}} \int [y + B]^n \, e^{-\frac{y^2}{2A}}\, dy$$

$$= \frac{1}{\sqrt{2\pi A}} \int \left[y^n + Bny^{n-1} + B^2 \frac{n(n-1)}{2} y^{n-2} + \ldots\right] e^{-\frac{y^2}{2A}}\, dy,$$

for which only the even powers of y are non-vanishing.

For n even:

$$\int \xi^n q(\xi, t, dt, x)\, d\xi = \frac{1}{\sqrt{2\pi A}} \int \left[y^n + B^2 \frac{n(n-1)}{2} y^{n-2} + \ldots\right] e^{-\frac{y^2}{2A}}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) (2A)^{n/2} \left[1 + \frac{nB^2}{(2A)} + \ldots\right]$$

$$= \frac{2^{n/2}}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) (\varepsilon a)^{n/2} (dt)^{n/2} + \text{terms of higher order in } (dt).$$

Thus, $\int \xi^n q(\xi, t, dt, x)\, d\xi = o(dt)$ for n even ≥ 4.

For n odd:

$$\int \xi^n q(\xi, t, dt, x)\, d\xi = \frac{1}{\sqrt{2\pi A}} \int \left[nBy^{n-1} + B^3 \frac{n(n-1)(n-2)}{6} y^{n-3} + \ldots\right] \cdot e^{-\frac{y^2}{2A}}\, dy$$

$$= \frac{2B}{\sqrt{\pi}} \Gamma\left(\frac{n+2}{2}\right) (2A)^{\frac{(n-1)}{2}} \left[1 + \frac{(n-1)B^2}{3(2A)} + \ldots\right]$$

$$= \frac{2^{(n+1)/2}}{\sqrt{\pi}} \Gamma\left(\frac{n+2}{2}\right) b \cdot (\varepsilon a)^{\frac{(n-1)}{2}} (dt)^{\frac{(n+1)}{2}} + \text{terms higher order in } (dt).$$

Thus,

$$\int \xi^n q(\xi, t, dt, x)\, d\xi = o(dt) \quad \text{for} \quad n \text{ odd} \geq 3;$$

and

$$\therefore \int \xi^n q(\xi, t, dt, x)\, d\xi = o(dt) \quad \text{for} \quad n \geq 3.$$

EXERCISE:

Suppose that there is no search. Let

$$\rho(x,t)dA(x) = \text{Prob}\{X(t)\in dA(x)\} .$$

What equation does ρ satisfy?

SOLUTION:

$$\rho(\underline{x},t+dt) = \int q(\underline{\xi},t,dt,\underline{x}-\underline{\xi})\rho(\underline{x}-\underline{\xi},t)d\underline{\xi}$$

$$= \int d\underline{\xi}\left[q\rho - \sum_i \xi_i \frac{\partial}{\partial x_i}(q\rho) + \frac{1}{2}\sum_{i,j}\xi_i\xi_j\frac{\partial^2}{\partial x_i\partial x_j}(q\rho)\right]$$

where $q\rho = q(\underline{\xi},t,dt,\underline{x})\rho(\underline{x},t)$

$$= \rho(\underline{x},t)\int d\underline{\xi}q - \sum_i \frac{\partial}{\partial x_i}\rho\int d\underline{\xi}\xi_i q + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\rho\int d\underline{\xi}\xi_i\xi_j q .$$

Thus,

$$\rho(\underline{x},t+dt)-\rho(\underline{x},t) = -\sum_i \frac{\partial}{\partial x_i}\rho[b_i(\underline{x},t)dt + o(dt)]$$

$$+\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\rho[ca_{ij}(\underline{x},t)dt + o(dt)].$$

Dividing by dt and taking the limit $dt\to 0$,

$$\frac{\partial\rho}{\partial t} + \sum_i \frac{\partial}{\partial x_i}(b_i\rho) = \frac{c}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}\rho) .$$

EXERCISE:

For deterministic target motion, show that

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(b_i f) - \phi f .$$

SOLUTION:

For deterministic target motion, $\varepsilon \to 0$ in the stochastic search equation with a_{ij} bounded.

$\therefore$ Search equation becomes

$$\frac{\partial f}{\partial t} = -\phi f - \sum_i \frac{\partial}{\partial x_i}(b_i f) \quad .$$

This result is also obtained by setting

$$q(\xi, t, dt, x) = \delta\left(\xi - b(x,t)dt\right) \quad .$$

EXERCISE:

Suppose that $b_i(x,t) = v_i(t)$. Show that

$$f(x,t;Z) = \rho_0\left(x - \int_0^t v(S)dS\right) \cdot \exp\left[-\int_0^t \psi\left(x - \int_S^t v(S')dS',\ S;Z\right)dS\right] .$$

Write down the integral that gives the probability of detection by time t .

SOLUTION:

If $b_i(x,t) = v_i(t)$, $\frac{\partial b_i}{\partial x_i} = 0$, and

$$f(x,t;Z) = \rho_0(x_0)_{traj\ x} \exp\left[-\int_0^t \left(\psi + \sum_i \frac{\partial b_i}{\partial x_i}\right) dS\right]$$

becomes

$$f(x,t;Z) = \rho_0(x_0)_{traj\ x} \exp\left[-\int_0^t \psi\left(x(S),\ S;\ Z\right)dS\right] .$$

Now $$x(S) = x_0 + \int_0^S v(S')dS' ,$$

and $$x_0 = x - \int_0^t v(S)dS = x - \int_0^t v(S')dS' .$$

Thus,

$$x(S) = x(t) - \int_0^t v(S')dS' + \int_0^S v(S')dS'$$

$$= x(t) - \int_0^t v(S')dS' - \int_S^0 v(S')dS'$$

$$= x(t) - \int_S^t v(S')dS' ,$$

and

$$f(x,t;Z) = \rho_0\left(x - \int_0^t v(S)dS\right)$$

$$\cdot \exp\left[-\int_0^t \psi\left(x - \int_S^t v(S')dS', S; Z\right) dS\right] .$$

$$\text{Prob\{detection by time } t\} = 1 - \text{Prob\{no detection\}}$$

$$= 1 - \int f(x,t;Z)dx$$

$$= 1 - \rho_0\left(x - \int_0^t v(S)dS\right)$$

$$\cdot \exp\left[-\int_0^t \psi\left(x - \int_S^t v(S')dS', S; Z\right)dS\right]dx .$$

EXERCISE:

Let $f(x,t) = w(x,t)e^{-\bar{\psi}t}$. Show that

$$\frac{\partial w}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} \delta_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial w}{\partial x_i}$$

and $w(x,0) = \rho_0(x)$.

SOLUTION:

Given $f(x,t) = w(x,t)e^{-\bar{\psi}t}$,

$$\frac{\partial f}{\partial t} = \frac{\partial w}{\partial t} e^{-\bar{\psi}t} - \bar{\psi} w(x,t) e^{-\bar{\psi}t}$$

$$= \left(\frac{\partial w}{\partial t}\right) e^{-\bar{\psi}t} - \bar{\psi} f(x,t).$$

For $a_{ij} = \delta_{ij}$ and b_i constant,

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij} f) - \sum_i \frac{\partial}{\partial x_i} (b_i f) - \psi f$$

becomes

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} \delta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial f}{\partial x_i} - \bar{\psi} f$$

$$= \left(\frac{\partial w}{\partial t}\right) e^{-\bar{\psi}t} - \bar{\psi} f.$$

Thus,

$$\left(\frac{\partial w}{\partial t}\right) e^{-\bar{\psi} t} = \frac{\varepsilon}{2} \sum_{i,j} \delta_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial f}{\partial x_i}$$

$$= \left[\frac{\varepsilon}{2} \sum_{i,j} \delta_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial w}{\partial x_i}\right] e^{-\bar{\psi} t} ,$$

and $$\therefore \frac{\partial w}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} \delta_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} - \sum_i b_i \frac{\partial w}{\partial x_i} .$$

Also, $$\rho_0(x) = f(x,0) = w(x,0)$$

EXERCISE:

In two dimensions, $w(x,t) = G(x-\xi,t)\rho_0(\xi)d\xi$, with

$$G(x-\xi,t) = \frac{1}{2\pi\varepsilon t}\exp\left[-\frac{(x_1-b_1t-\xi_1)^2+(x_2-b_2t-\xi_2)^2}{2\varepsilon t}\right].$$

Assume

$$\rho_0(x) = \frac{1}{2\pi\sigma^2}e^{-\left[\frac{(x_2)^2+(x_1)^2}{2\sigma^2}\right]}.$$

Find $w(x,t)$. HINT: Gaussians are closed under convolution.

SOLUTION:

$$w(x,t) = \int G(x-\xi,t)\rho_0(\xi)d\xi$$

$$= \int d\xi_1 d\xi_2 \frac{1}{2\pi\varepsilon t}\exp\left[\frac{(x_1-b_1t-\xi_1)^2+(x_2-b_2t-\xi_2)^2}{2\varepsilon t}\right]$$

$$\cdot \frac{1}{2\pi\sigma^2}\exp\left[-\frac{(\xi_2)^2+(\xi_1)^2}{2\sigma^2}\right]$$

$$= \frac{1}{4\pi^2\sigma^2\varepsilon t}\int\exp\left[-\frac{(x_1-b_1t-\xi_1)^2}{2\varepsilon t}-\frac{(\xi_1)^2}{2\sigma^2}\right]d\xi_1$$

$$\cdot\int\exp\left[-\frac{(x_2-b_2t-\xi_2)^2}{2\varepsilon t}-\frac{(\xi_2)^2}{2\sigma^2}\right]d\xi_2 .$$

Now,

$$(x_1-b_1t-\xi_1)^2 = (\xi_1)^2 - 2(x_1-b_1t)\xi_1 + (x_1-b_1t)^2$$

and

$$\frac{(x_1-b_1t-\xi_1)^2}{2\varepsilon t} + \frac{(\xi_1)^2}{2\sigma^2} = \frac{2(\sigma^2+\varepsilon t)(\xi_1)^2-4\sigma^2(x_1-b_1t)\xi_1+2\sigma^2(x_1-b_1t)^2}{4\sigma^2\varepsilon t}$$

$$= \frac{\left[\sqrt{2(\sigma^2+\varepsilon t)}\ \xi_1 - \frac{\sqrt{2}\ \sigma^2(x_1-b_1t)}{\sqrt{(\sigma^2+\varepsilon t)}}\right]^2 + \left[\frac{2\sigma^2\varepsilon t(x_1-b_1t)^2}{(\sigma^2+\varepsilon t)}\right]}{4\sigma^2\varepsilon t}.$$

Consequently,

$$\int \exp\left[-\frac{(x_1-b_1t-\xi_1)^2}{2\varepsilon t} - \frac{(\xi_1)^2}{2\sigma^2}\right] d\xi_1$$

$$= \int \exp\left\{-\left[\frac{\sqrt{2}\ (\sigma^2+\varepsilon t)\xi_1-\sqrt{2}\ \sigma^2(x_1-b_1t)}{\sqrt{4\sigma^2\varepsilon t}\ \sqrt{(\sigma^2+\varepsilon t)}}\right]^2 - \left[\frac{(x_1-b_1t)^2}{2(\sigma^2+\varepsilon t)}\right]\right\} d\xi_1$$

$$= \exp\left[-\frac{(x_1-b_1t)^2}{2(\sigma^2+\varepsilon t)}\right] \int d\xi_1 \exp\left\{-\left[\frac{\sqrt{2}\ (\sigma^2+\varepsilon t)\xi_1-\sqrt{2}\ \sigma^2(x_1-b_1t)}{\sqrt{4\sigma^2\varepsilon t}\ \sqrt{(\sigma^2+\varepsilon t)}}\right]^2\right\}$$

$$= \exp\left[-\frac{(x_1-b_1t)^2}{2(\sigma^2+\varepsilon t)}\right] \frac{\sqrt{2\sigma^2\varepsilon t}}{\sqrt{(\sigma^2+\varepsilon t)}} \int e^{-y^2}\, dy$$

$$= \sqrt{\frac{2\pi\sigma^2\varepsilon t}{(\sigma^2+\varepsilon t)}} \exp\left[-\frac{(x_1-b_1t)^2}{2(\sigma^2+\varepsilon t)}\right].$$

Similarly,

$$\int \exp\left[-\frac{(x_2-b_2t-\xi_2)^2}{2\varepsilon t} - \frac{(\xi_2)^2}{2\sigma^2}\right] d\xi_2 = \sqrt{\frac{2\pi\sigma^2\varepsilon t}{(\sigma^2+\varepsilon t)}} \exp\left[-\frac{(x_2-b_2t)^2}{2(\sigma^2+\varepsilon t)}\right],$$

and

$$w(x,t) = \frac{1}{4\pi^2\sigma^2\varepsilon t} \int \exp\left[-\frac{(x_1b_1t-\xi_1)^2}{2\varepsilon t} - \frac{(\xi_1)^2}{2\sigma^2}\right] d\xi_1$$

$$\cdot \int \exp\left[-\frac{(x_2-b_2t-\xi_2)^2}{2\varepsilon t} - \frac{(\xi_2)^2}{2\sigma^2}\right] d\xi_2$$

$$= \frac{1}{2\pi(\sigma^2+\varepsilon t)} \exp\left[-\frac{(x_1-b_1t)^2+(x_2-b_2t)^2}{2(\sigma^2+\varepsilon t)}\right].$$

EXERCISE:

Consider the one-dimensional equation,

$$w(x,t) = \frac{1}{\sqrt{2\pi\varepsilon t}} \int_{\ell}^{\ell} \bar{\rho}(\xi) e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi \quad .$$

If $\bar{\rho}(\xi) = 1/2\ell$, use integration by parts to derive an expansion for $w(x,t)$.

SOLUTION:

Substituting in for $\bar{\rho}(\xi)$ gives

$$w(x,t) = \frac{1}{\sqrt{2\pi\varepsilon t}} \frac{1}{2\ell} \int_{\ell}^{\ell} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi$$

$$= \frac{1}{\sqrt{2\pi\varepsilon t}} \left\{ \int_{\infty}^{\ell} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi - \int_{\infty}^{-\ell} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi \right\} .$$

$$= \frac{1}{\sqrt{2\pi\varepsilon t}} \left\{ \int_{-\infty}^{\infty} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi \right.$$

$$\left. - \int_{\ell}^{\infty} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi - \int_{-\infty}^{-\ell} e^{-(x-bt-\xi)^2/2\varepsilon t} \, d\xi \right\}$$

$$= \frac{1}{\sqrt{2\pi\varepsilon t}}\left\{\sqrt{2\pi\varepsilon t} - \int_{\ell}^{\infty} e^{-(x-bt-\xi)^2/2\varepsilon t}\,d\xi - \int_{-\infty}^{\ell} e^{-(x-bt-\xi)^2/2\varepsilon t}\,d\xi\right\}$$

$$= \frac{1}{\sqrt{2\pi\varepsilon t}}\left\{\sqrt{2\pi\varepsilon t} - \int_{\ell}^{\infty} +\frac{\varepsilon t}{(x-bt-\xi)}\,\frac{d}{d\xi}\left[e^{-(x-bt-\xi)^2/2\varepsilon t}\right]d\xi\right.$$

$$\left. - \int_{-\infty}^{\ell} +\frac{\varepsilon t}{(x-bt-\xi)}\,\frac{d}{d\xi}\left[e^{-(x-bt-\xi)^2/2\varepsilon t}\right]d\xi\right\} .$$

Now integrate by parts

$$w(x,t) = \frac{1}{\sqrt{2\pi\varepsilon t}}\left\{\sqrt{2\pi\varepsilon t} - \frac{\varepsilon t}{(x-bt-\ell)}\,e^{-(x-bt-\ell)^2/2\varepsilon t}\right.$$

$$+ \int_{-\infty}^{\ell} \frac{\varepsilon t}{(x-bt-\xi)^2}\,e^{-(x-bt-\xi)^2/2\varepsilon t}\,d\xi - \frac{\varepsilon t}{(x-bt+\ell)}\,e^{-(x-bt+\ell)^2/2\varepsilon t}$$

$$\left. + \int_{-\infty}^{\ell} \frac{\varepsilon t}{(x-bt-\xi)^2}\,e^{-(x-bt-\xi)^2/2\varepsilon t}\,d\xi\right\} .$$

Now repeat the process.

EXERCISE:

For solution of the search equation taking the form,

$$f(x,t) = g(x,t;\varepsilon)\exp[-\phi(x,t;\varepsilon)] \ ,$$

pick $g(x,t;\varepsilon)$, $\phi(x,t;\varepsilon)$ and find the b^i necessary to satisfy the search equation.

SOLUTION:

This is an audience participation problem. It would defeat the purpose to give a worked example.

EXERCISE:

Using the ray ansatz,

$$f(x,t;z) = \left\{ \sum_{k=0}^{\infty} \varepsilon^k g_k(x,t;z) \right\} e^{-\phi(x,t)/\varepsilon} .$$

The equation for $\phi(x,t)$ is obtained by setting the coefficient of $\frac{1}{\varepsilon} e^{-\phi/\varepsilon}$ obtained from the search equation equal to zero. Show that this equation is

$$\frac{\partial \phi}{\partial t} + \sum_i b_i \frac{\partial \phi}{\partial x^i} + \frac{1}{2} \sum_{i,j} a^{ij} \left(\frac{\partial \phi}{\partial x^i} \right) \left(\frac{\partial \phi}{\partial x^j} \right) = 0 .$$

SOLUTION:

The search equation,

$$\frac{\partial f}{\partial t} = -\psi f - \sum_i \frac{\partial}{\partial x_i} (b_i f) + \frac{\varepsilon}{2} \sum_{i,j} \frac{\partial^2 (a_{ij} f)}{\partial x_i \partial x_j} ,$$

is written equivalently as

$$\frac{\partial f}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_i \left\{ b_i - \varepsilon \sum_j \left(\frac{\partial a_{ij}}{\partial x_j} \right) \right\} \left(\frac{\partial f}{\partial x_i} \right)$$

$$- \left\{ \psi + \sum_i \left[\left(\frac{\partial b_i}{\partial x_i} \right) - \frac{\varepsilon}{2} \sum_j \left(\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} \right) \right] \right\} f .$$

For $$f(x,t;z) = \left\{ \sum_{k=0}^{\infty} \varepsilon^k g_k(x,t;z) \right\} e^{-\phi(x,t)/\varepsilon} ,$$

$$\frac{\partial f}{\partial t} = \left\{ \sum_{k=0}^{\infty} \left[\varepsilon^k \frac{\partial g_k}{\partial t} - \varepsilon^{k-1} \left(\frac{\partial \phi}{\partial t} \right) g_k \right] \right\} e^{-\phi/\varepsilon} ,$$

$$\frac{\partial f}{\partial x_i} = \left\{ \sum_{k=0}^{\infty} \left[\varepsilon^k \left(\frac{\partial g_k}{\partial x_i} \right) - \varepsilon^{k-1} \left(\frac{\partial \phi}{\partial x_i} \right) g_k \right] \right\} e^{-\phi/\varepsilon} ,$$

and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \left[\sum_{k=0}^{\infty} \left\{ \varepsilon^k \left(\frac{\partial^2 g_k}{\partial x_i \partial x_j} \right) - \varepsilon^{k-1} \left[2 \left(\frac{\partial g_k}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) + g_k \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) \right] + \varepsilon^{k-2} g_k \left(\frac{\partial \phi}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) \right\} \right] e^{-\phi/\varepsilon} ,$$

and the search equation becomes

$$\left\{ \sum_{k=0}^{\infty} \left[\varepsilon^k \left(\frac{\partial g_k}{\partial t} \right) - \varepsilon^{k-1} \left(\frac{\partial \phi}{\partial t} \right) g_k \right] \right\} e^{-\phi/\varepsilon}$$

$$= \left[\sum_{i,j} \frac{a_{ij}}{2} \sum_{k=0}^{\infty} \left\{ \varepsilon^{k+1} \left(\frac{\partial^2 g_k}{\partial x_i \partial x_j} \right) - \varepsilon^k \left[2 \left(\frac{\partial g_k}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) + g_k \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) \right] + \varepsilon^{k-1} g_k \left(\frac{\partial \phi}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) \right\} \right] e^{-\phi/\varepsilon}$$

$$- \left\{ \sum_i \left[b^i - \varepsilon \sum_j \left(\frac{\partial a_{ij}}{\partial x_j} \right) \right] \sum_{k=0}^{\infty} \left[\varepsilon^k \left(\frac{\partial g_k}{\partial x_i} \right) - \varepsilon^{k-1} \left(\frac{\partial \phi}{\partial x_i} \right) g_k \right] \right\} e^{-\phi/\varepsilon}$$

$$- \left[\left\{ \psi + \sum_i \left[\left(\frac{\partial b_i}{\partial x_i} \right) - \frac{\varepsilon}{2} \sum_j \left(\frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} \right) \right] \right\} \sum_{k=0}^{\infty} \varepsilon^k g_k \right] e^{-\phi/\varepsilon} .$$

One now equates terms with the same power of ε . The equation for $\phi(x,t)$ is obtained by looking at terms involving $1/\varepsilon$:

$$-\left(\frac{\partial \phi}{\partial t}\right) g_0 \frac{1}{\varepsilon} e^{-\phi/\varepsilon} = \sum_{i,j} \frac{a_{ij}}{2} g_0 \left(\frac{\partial \phi}{\partial x_i}\right)\left(\frac{\partial \phi}{\partial x_j}\right) \frac{1}{\varepsilon} e^{-\phi/\varepsilon}$$

$$+ \sum_i b_i \left(\frac{\partial \phi}{\partial x_i}\right) g_0 \frac{1}{\varepsilon} e^{-\phi/\varepsilon} \quad .$$

This yields

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \sum_{i,j} a_{ij} \left(\frac{\partial \phi}{\partial x_i}\right)\left(\frac{\partial \phi}{\partial x_j}\right) + \sum_i b_i \left(\frac{\partial \phi}{\partial x_i}\right) = 0 \ ,$$

the desired result.

EXERCISE:

What equation does $g_0(x,t;z)$ satisfy?

SOLUTION:

This is obtained by looking at the terms involving ε^0:

$$\left[\left(\frac{\partial g_0}{\partial t}\right) - \left(\frac{\partial \phi}{\partial t}\right) g_1\right] e^{-\phi/\varepsilon} = \sum_{i,j} \frac{a_{ij}}{2} \left\{ g_1 \left(\frac{\partial \phi}{\partial x_i}\right) \frac{\partial \phi}{\partial x_j} \right.$$

$$\left. - \left[2\left(\frac{\partial g_0}{\partial x_i}\right)\left(\frac{\partial \phi}{\partial x_j}\right) + g_0 \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right)\right] \right\} e^{-\phi/\varepsilon}$$

$$- \sum_i b_i \left[\left(\frac{\partial g_0}{\partial x_i} \right) - \left(\frac{\partial \phi}{\partial x_i} \right) g_1 \right] e^{-\phi/\varepsilon}$$

$$- \sum_{i,j} \left[\left(\frac{\partial a_{ij}}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) g_0 \right] e^{-\phi/\varepsilon}$$

$$- \left[\psi + \sum_i \left(\frac{\partial b_i}{\partial x_i} \right) \right] g_0 \; e^{-\phi/\varepsilon} \quad .$$

Upon rearranging terms, one obtains

$$\frac{\partial g_0}{\partial t} + \sum_{i,j} \left\{ \frac{a_{ij}}{2} \left[2 \left(\frac{\partial g_0}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) + g_0 \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) \right] + \left[\left(\frac{\partial a_{ij}}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) g_0 \right] \right\}$$

$$+ \sum_i \left\{ b_i \left(\frac{\partial g_0}{\partial x_i} \right) + g_0 \left(\frac{\partial b_i}{\partial x_i} \right) \right\} + \psi g_0 \quad = \quad 0 \; ,$$

since $g_1 \left\{ \frac{\partial \phi}{\partial t} + \sum_{i,j} \frac{a_{ij}}{2} \left(\frac{\partial \phi}{\partial x_i} \right) \left(\frac{\partial \phi}{\partial x_j} \right) + \sum_i b_i \left(\frac{\partial \phi}{\partial x_i} \right) \right\}$ vanishes.

(The term in braces is the left hand side of the equation just obtained for ϕ .)

EXERCISE:

Assume $a_{ij} = \delta_{ij}$ and $b_i(x,t) = b_i$, a constant. Write and solve the ray equations.

SOLUTION:

The ray equations are the following:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \text{with} \quad x_i(0) = x_{i0}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad \text{with} \quad p_i(0) = p_{i0} \ ,$$

where

$$H(x,p) = \sum_i b_i p_i + \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j \ .$$

For $a_{ij} = \delta_{ij}$,

$$H(x,p) = \sum_i (b_i p_i + \frac{1}{2} p_i^2) \ .$$

Now

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = b_i + p_i$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = 0 \ .$$

The second of these yields the solution, $p_i(t) = p_{i0}$.

Then, integration of the first yields

$$x_i(t) - x_i(0) = \int_0^t (b_i+p_i)dt' = (b_i+p_{i0})t ,$$

and consequently,

$$x_i(t) = x_{i0} + (b_i+p_{i0})t$$

These rays are simply straight lines.

EXERCISE:

Assume $a_{ij} = \delta_{ij}$ and that $b_i(x,t)$ is a function of t only, i.e., $b_i(x,t) = v_i(t)$. Write the ray equation.

SOLUTION:

$$H(x,p) = \sum_i b_i p_i + \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j \quad .$$

For $a_{ij} = \delta_{ij}$ and $b_i(x,t) = v_i(t)$,

$$H(x,p) = \sum_i \left[v_i(t) p_i + \frac{1}{2} p_i^2 \right] \quad .$$

RAY EQUATIONS

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} = p_i(t) + v_i(t) & x_i(0) = x_{i0} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i} = 0 & p_i(t) = p_i(0) = p_{i0} \end{cases} \quad .$$

Integrating the equation for x^i ,

$$x_i(t) - x_i(0) = \int_0^t \left[p_i(t') + v_i(t') \right] dt' \quad ,$$

and consequently,

$$x_i(t) = x_{i0} + p_{i0} t + \int_0^t v_i(t') dt'$$

EXERCISE:

What are the ray equations for general $a_{ij}(x,t)$ and $b_i(x,t)$?

SOLUTION:

For $a_{ij}(x,t)$ and $b_i(x,t)$ generally,

$$H(x,p) = \sum_i b_i(x,t)p_i(t) + \frac{1}{2}\sum_{i,j} a_{ij}(x,t)p_i(t)p_j(t) \quad ,$$

and consequently,

$$\frac{\partial H}{\partial p_i} = b_i(x,t) + \sum_j a_{ij}(x,t)p_j(t)$$

and

$$\frac{\partial H}{\partial x_i} = \sum_i p_i(t)\left(\frac{\partial b_i}{\partial x_i}\right) + \frac{1}{2}\sum_{i,j} p_i(t)p_j(t)\left(\frac{\partial a_{ij}}{\partial x_i}\right) \quad .$$

$\therefore$ The ray equations can be written

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = b_i(x,t) + \sum_j a_{ij}(x,t)p_j(t) \quad ; \qquad x_i(0) = x_{i0}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_i p_i(t)\left(\frac{\partial b_i}{\partial x_i}\right) - \frac{1}{2}\sum_{i,j} p_i(t)p_j(t)\left(\frac{\partial a_{ij}}{\partial x_i}\right) \quad .$$

EXERCISE:

Suppose $\rho_0(x) = \frac{1}{2\pi\sigma^2} e^{-[(x_1)^2+(x_2)^2]/2\sigma^2}$.

What are ϕ and h_k $(k = 0, 1, 2, \ldots)$ if ρ_0 is written in the form

$$\rho_0(x) = e^{-\phi(x)/\varepsilon} \sum_{k=0}^{\infty} h_k(x)\varepsilon^k \quad ?$$

SOLUTION:

First, equate exponentials:

$$-\phi(x)/\varepsilon = -\left[(x_1)^2+(x_2)^2\right]/2\sigma^2 .$$

This yields

$$\phi(x) = \frac{\varepsilon\left[(x_1)^2+(x_2)^2\right]}{2\sigma^2} .$$

Next, equate the remaining terms:

$$\sum_{k=0}^{\infty} h_k(x)\varepsilon^k = \frac{1}{2\pi\sigma^2} .$$

If the $h_k(x)$ are assumed to be zeroth order in ε, the above equation gives

$$h_0(x) = \frac{1}{2\pi\sigma^2} \quad \text{and} \quad h_k(x) = 0 \quad \text{for} \quad k \neq 0 .$$

EXERCISE:

The Lagrangian $L\left(x, \frac{dx}{dt}\right)$ is defined such that

$$L\left(x, \frac{dx}{dt}\right) + H(x,p) = \sum_i \frac{dx_i}{dt} \cdot p_i \quad .$$

Show that

$$L\left(x, \frac{dx}{dt}\right) = \frac{1}{2} \sum_{i,j} \left(\frac{dx_i}{dt} - b_i\right) \left(a_{ij}\right)^{-1} \left(\frac{dx_i}{dt} - b_i\right) \quad ,$$

where $\left(a_{ij}\right)^{-1}$ is the i,j^{th} element of the inverse of the matrix $[a_{ij}]$.

SOLUTION:

$$H(x,p) = \sum_i b_i p_i + \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j$$

and

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = b_i + \sum_j a_{ij} p_j \quad .$$

Now,

$$\frac{dx_i}{dt} - b_i = \sum_j a_{ij} p_j$$

can be written as

$$\left(\frac{dx}{dt} - \underline{b}\right)_i = \left([\underline{a}_{ij}] \cdot \underline{p}\right)_i \quad ,$$

where $\left(\frac{d\underline{x}}{dt} - \underline{b}\right)_i$ is the i^{th} row of the nx1 column matrix, the elements of which are the components of the vector $\left(\frac{d}{dt}\underline{x} - \underline{b}\right)$. $[\underline{\underline{a}}_{ij}] \cdot \underline{p}$ is the product of the nxm matrix with elements a_{ij} and the column matrix (nx1) whose elements are the components of the vector $\underline{p}$.

Since

$$\left(\frac{d\underline{x}}{dt} - \underline{b}\right) = [\underline{\underline{a}}_{ij}] \cdot \underline{p} \ ,$$

$$\underline{p} = [\underline{\underline{a}}_{ij}]^{-1} \cdot \left(\frac{d\underline{x}}{dt} - \underline{b}\right) \ ,$$

where $[\underline{\underline{a}}_{ij}]^{-1}$ is the inverse of the matrix $[\underline{\underline{a}}_{ij}]$.

Now

$$p_k = \left([\underline{\underline{a}}_{ij}]^{-1} \cdot \left(\frac{d\underline{x}}{dt} - \underline{b}\right)\right)_k$$

$$= \sum_{\ell} \left([\underline{\underline{a}}_{ij}]^{-1}\right)_{k\ell} \left(\frac{dx_\ell}{dt} - b_\ell\right) \ ,$$

where $\left([\underline{\underline{a}}_{ij}]^{-1}\right)_{k\ell}$ is the k,ℓ^{th} element of the matrix that is the inverse of the matrix $[\underline{\underline{a}}_{ij}]$.

Turning to the Lagrangian,

$$L\left(x, \frac{dx}{dt}\right) = \sum_i \frac{dx_i}{dt} \cdot p_i - H(x,p)$$

$$= \underbrace{\sum_i b_i p_i + \sum_{i,j} a_{ij} p_j p_i}_{\sum_i \frac{dx_i}{dt} \cdot p_i} - \underbrace{\left[\sum_i b_i p_i + \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j\right]}_{H(x,p)}$$

$$= \frac{1}{2} \sum_{i,j} a_{ij} p_i p_j$$

$$= \frac{1}{2} (\underline{p})^{Tr} \cdot [\underline{\underline{a}}_{ij}] \cdot (\underline{p}) \quad ,$$

where $(\underline{p})^{Tr}$ is the 1xn row matrix obtained by taking the transpose of the nx1 column matrix $(\underline{p})$.

$$(\underline{p})^{Tr} = \left(\frac{d\underline{x}}{dt} - \underline{b}\right)^{Tr} \left([\underline{\underline{a}}_{ij}]^{-1}\right)^{Tr}$$

$$= \left(\frac{d\underline{x}}{dt} - \underline{b}\right)^{Tr} \left([\underline{\underline{a}}_{ij}]^{Tr}\right)^{-1}$$

$$= \left(\frac{d\underline{x}}{dt} - \underline{b}\right)^{Tr} [\underline{\underline{a}}_{ij}]^{-1} \quad .$$

Now,

$$(\underline{p})^{Tr}[\underline{\underline{a}}_{ij}]\cdot(\underline{p}) = \left(\frac{d\underline{x}}{dt} - \underline{b}\right)^{Tr}[a_{ij}]^{-1}[a_{ij}][a_{ij}]^{-1}\left(\frac{d\underline{x}}{dt} - \underline{b}\right)$$

$$= \left(\frac{d\underline{x}}{dt} - \underline{b}\right)^{Tr}[\underline{\underline{a}}_{ij}]^{-1}\left(\frac{d\underline{x}}{dt} - \underline{b}\right)$$

$$= \sum_{k\ell}\left(\frac{dx_k}{dt} - b_k\right)\left([\underline{\underline{a}}_{ij}]^{-1}\right)_{k\ell}\left(\frac{dx_\ell}{dt} - b_\ell\right)$$

$$= \sum_{k\ell}\left(\frac{dx_k}{dt} - b_k\right)(a_{ij})^{-1}\left(\frac{dx_\ell}{dt} - b_\ell\right) \quad ,$$

if one defines $(a_{ij})^{-1} \equiv \left([\underline{\underline{a}}_{ij}]^{-1}\right)_{k\ell}$ to be the k,ℓ^{th} element of the inverse of the matrix with elements a_{ij} .

Finally,

$$L\left(x,\frac{dx}{dt}\right) = \frac{1}{2}\sum_{i,j} a_{ij}p_ip_j = \frac{1}{2}(\underline{p})^{Tr}\cdot[\underline{\underline{a}}_{ij}]\cdot(\underline{p})$$

$$= \frac{1}{2}\sum_{i,j}\left(\frac{dx_i}{dt} - b_i\right)(a_{ij})^{-1}\left(\frac{dx_j}{dt} - b_j\right) \quad ,$$

which was to be shown.

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